Riemannian Geometry Notes

Abstract

These notes are from Math 7120 taught in Spring 2026 by Professor Ioana Suvaina and prepared by Brian Morton.

Contents

1 Review of Smooth Manifolds

1.1 Introduction to Manifolds

Definition 1.1. A **topological manifold** of dimension n is a set M such that

1. M is a Hausdorff space
2. at each point in M, there is a neighborhood U with is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 1.2. If the charts (U, φ) and (V, ψ) are such that $U \cap V = \emptyset$, the composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called the **transition map** from φ to ψ, We say that two charts (U, φ) and (V, ψ) are C^∞ **comparable** if their compositions $\varphi \circ \phi^{-1}$ and $\phi \circ \varphi^{-1}$ are also C^∞.

Definition 1.3. A **differentiable structure** on a topological manifold M is given by a family $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of coordinate neighborhoods such that

1. $\bigcup_\alpha U_\alpha = M$
2. for all $\alpha, \beta \in I$, the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are C^∞ comparable
3. Any chart (V, ψ) comparable with $(U_\alpha, \varphi_\alpha)$ is itself in $\mathcal{U}$.

A C^∞ manifold is a topological manifold together with a C^∞ differentiable structure on the manifold. If $(M,\mathcal{U})$ satisfies properties 1 and 2, we call this an **atlas** on M. If $(M,\mathcal{U})$ satisfies properties 1, 2, and 3, we call this a **maximal atlas** on M.

Remark. Two atlases are comparable if their union forms an atlas.

Example 1.4. We provide some basic examples of smooth manifolds that we will reference throughout these notes.

1. The Cartesian plane: $\mathbb{R}^n$
2. The $n-$sphere: S^n
3. The projective plane: $\mathbb{R}P^n := [\mathbb{R}^n/\{0\}] / \sim$ where the equivalence relation is given by $(x_1, ..., x_{n+1}) \sim (tx_1, ..., tx_{n+1})$ for $0 \neq t \in \mathbb{R}$.
4. In dimension two, we have two examples:
   (a) The torus: $T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$
   (b) A genus g Riemannian surface: Σ_g
5. The space of matrices with real values: $\mathcal{M}_{n\times n}(\mathbb{R})$

6. The general linear group: $GL(n, \mathbb{R}) \coloneqq \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) \mid \det(A) \neq 0\}$
7. The space of orthogonal matrices: $O(n) \coloneqq \{A \in \mathcal{M}_{n\times n}(\mathbb{R}) \mid AA^T = id\}$. Note that this manifold is not compact. This presents difficulties in further analysis. Accordingly, we can consider compact subgroups of the general linear group.
8. The symmetric orthogonal group: $SO(n) \coloneqq \{A \in O(n) \mid \det(A) = 1\}$. Note that this manifold is orientation preserving.
9. Open subsets of a manifold.
10. Closed subsets are not necessarily smooth manifolds.

Remark. 1. Not every topological manifold admits a smooth manifold.

2. There are topological manifolds that admit infinitely many nonequivalent smooth manifolds. We discuss some in the following examples.

   Example 1.5. Consider the following sum of complex projective planes:

$$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2} \# \cdots \# \overline{\mathbb{C}P^2}$$

   where $\overline{\mathbb{C}P^2}$ is the complex projective space with a reversed orientation. The above space has at least two distinct smooth structures: one with positive curvature and another with negative curvature. If we want to consider infinitely many nonequivalent smooth structures, we need to consider smooth invariants and consider a metric.

Proposition 1.6. *Let M be a topological manifold. Then, the following hold:*

1. *Every smooth atlas $(M, \mathcal{U})$ is contained in a unique maximal smooth atlas.*
2. *Two atlases determine the same smooth if and only if their union is a smooth atlas.*

Example 1.7. Consider the sphere S^2. We present the following two atlases:
Atlas 1: Let N and S represent the north and south poles, respectively, of S^2. Consider the following two open sets

$$U_N \coloneqq S^2 \setminus \{N\} \qquad U_S \coloneqq S^2 \setminus \{S\}$$

and the following two charts

$$\pi_N : U_N \to \mathbb{R}^2$$
$$\pi_S : U_S \to \mathbb{R}^2$$

be the typical stereographic projections. It is an exercise to check that $\pi_N \circ \pi_S^{-1} (\pi_S \circ \pi_N^{-1})$ is smooth.
Atlas 2: Consider the following open sets of S^2

$$U_{x,+} \coloneqq \{(x, y, z) \in S^2 \mid x > 0\} \qquad U_{x,-} \coloneqq \{(x, y, z) \in S^2 \mid x < 0\}$$
$$U_{y,+} \coloneqq \{(x, y, z) \in S^2 \mid y > 0\} \qquad U_{y,-} \coloneqq \{(x, y, z) \in S^2 \mid y < 0\}$$

$$U_{z,+} \coloneqq \{(x,y,z) \in S^2 \mid z > 0\} \qquad U_{z,-} \coloneqq \{(x,y,z) \in S^2 \mid z < 0\}$$

and the following charts given by

$$\pi_{x,\pm} : U_{x,\pm} \to D^2 = \{(y,z) \in \mathbb{R}^2 \mid y^2 + z^2 < 1\}$$
$$\pi_{y,\pm} : U_{y,\pm} \to D^2 = \{(x,z) \in \mathbb{R}^2 \mid x^2 + z^2 < 1\}$$
$$\pi_{z,\pm} : U_{z,\pm} \to D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 < 1\}$$

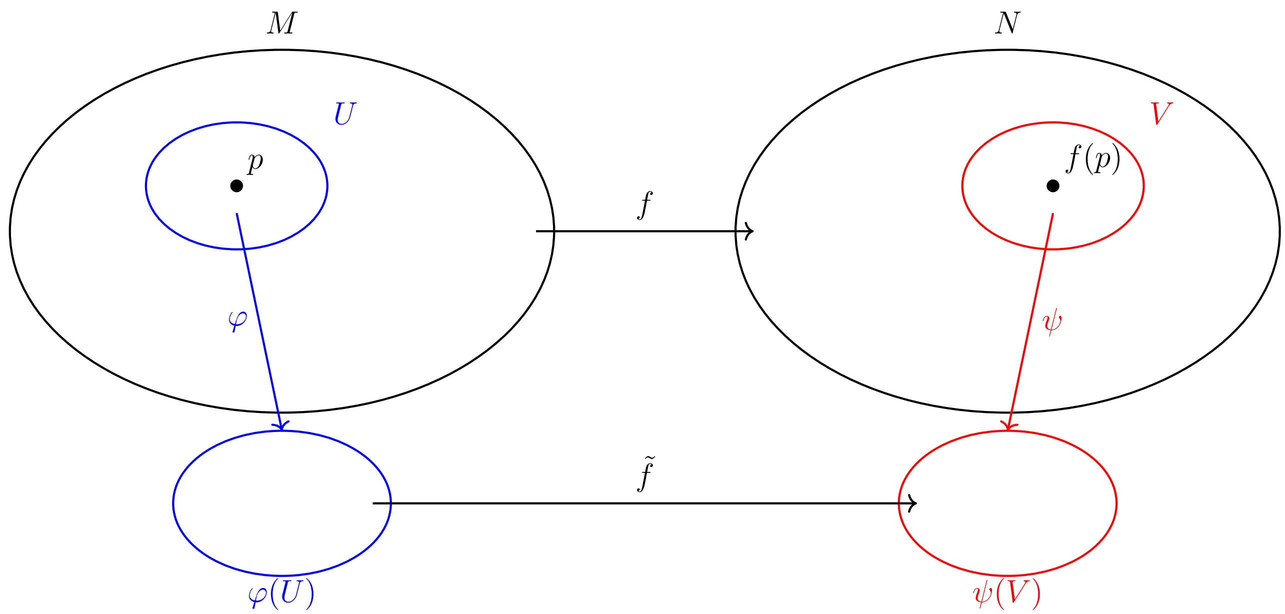
To see if these two atlases admit the same smooth structure, we have to check if the composition of maps is smooth.

1.2 Smooth Functions

Definition 1.8. Let M and N be smooth manifolds. We say that a function $f : M \to N$ is **smooth** if for any $p \in M$, there exists local charts (U, φ) and (V, ψ) that contain p and $f(p)$, respectively, such that the composition $\psi \circ f \circ \varphi^{-1}$ is smooth. We set the above composition as $\tilde{f}$. Define the space $C^\infty(M)$ as

$$C^\infty(M) \coloneqq \{f : M \to \mathbb{R} \mid f \text{ is smooth}\}. \tag{1}$$

It follows that $C^\infty(M)$ is determined and determines the smooth structure on M.



We now consider how two manifolds are considered as *equal* in the differential geometry sense.

Definition 1.9. Let M and N be two smooth manifolds and let $f : M \to N$ be smooth. We say that f is a **diffeomorphism** if the following conditions are true

1. f is bijective
2. f is smooth
3. f^{-1} is smooth

We say that M and N are **diffeomorphic** if f is a diffeomorphism.

1.3 Tangent Vectors and the Tangent Space

We begin with a discussion of tangent vectors. There are a few ways to introduce these objects. We provide an overview of each and outline why each definition is equivalent.

1. **Definition 1: Geometric Approach**: Let M be a smooth manifold and fix $p \in M$. A **tangent vector** at p is given by an equivalence class of differenaible curves $c : (-\epsilon, \epsilon) \to M$ with $c(0) = p$ where $c \sim \gamma$ if and only if

$$\frac{d}{dt}(\varphi \circ c)\big|_{t=0} = \frac{d}{dt}(\varphi \circ \gamma)\big|_{t=0}$$

for a local chart (U, φ) containing p. Note that the definition does not require *every* chart. This is because our definition is independent of the choice of charts. Indeed, if (V, ψ) is a different chart, then we have

$$\frac{d}{dt}(\psi \circ c)\big|_{t=0} = \frac{d}{dt}(\psi \circ \gamma)\big|_{t=0}.$$

However, using the chain rule, we have that $\frac{d}{dt}(\psi \circ c)\big|_{t=0} = D(\psi \circ \varphi^{-1})\big|_{\varphi(p)} \frac{d}{dt}(\varphi \circ c)\big|_{t=0}$. As $D(\psi \circ \varphi^{-1})\big|_{\varphi(p)}$ is invertible, we see that the chart is independent of the above equivalence relation.

2. **Definition 2: Local Charts**: A tangent vector at p is given by the equivalence class (U, φ, a) where $(U, \varphi, a) \sim (V, \psi, b)$ if and only if $b = D\left(\psi \circ \varphi^{-1}\right) \cdot a$. Consider two charts: (U, φ) and (V, ψ). Let $T_{\varphi(p)}(\mathbb{R}^n) \ni x_U = a_i \partial_{x_i}$ and $T_{\psi(p)}(\mathbb{R}^n) \ni x_V = b_i \partial_{y_i}$ where $\frac{\partial}{\partial x_i} = \frac{d}{dt}(0, ..., 0, 1, 0, ..., 0)\big|_{t=0}$. How do we connect the two tangent spaces? We have that $b_i \partial_{y_i} = D(\psi \circ \varphi^{-1})_{\varphi(p)} a_i \partial_{x_i}$ where $D(\psi \circ \varphi^{-1})_{\varphi(p)} = \left(\frac{\partial y_i}{\partial x_j}\right)_{\varphi(p)}$.

   To show that the equivalence between our first two definitions, we have the following table.

Definition 1		**Definition 2**
$[c]$	$\longrightarrow$	$a = \frac{d}{dt}(\varphi \circ c)\Big\|_{t=0}$
$c(t) = \varphi^{-1}(t \cdot (a_1, ..., a_n) + \varphi(p))$	$\longleftarrow$	given a $\in T_{\varphi(p)}\mathbb{R}^n$

3. **Definition 3: Vectors as Derivations**: Let M be a smooth manifold and fix $p \in M$. A vector X at p is a derivation on $C^\infty(M)$ meaning that

$$X : C^\infty(M) \to M$$

such that

   (a) X is linear: $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$ for all $f, g \in C^\infty(M)$ and $\alpha, \beta \in \mathbb{R}$

(b) X obeys the Leinitz rule: $X(fg) = gX(f) + fX(g)$ for all $f, g \in C^\infty(M)$.

TO show the equivalence between definition one and definition 3, we again have the following table.

Definition 1		**Definition 3**
$[c] \ni c$	$\longrightarrow$	$X_c : C^\infty(M) \to \mathbb{R}$
$c(t) = \varphi^{-1}(t \cdot (a_1, ..., a_n) + \varphi(p))$	$\longleftarrow$	given a $\in T_{\varphi(p)}\mathbb{R}^n$

4. **Definition 4: Hybrid Definition**: A tangent vector X to the manifold X at p is a function

$$c'(0) : C^\infty(M) \to \mathbb{R}$$

given by a differential curve

$$\begin{aligned} &c : (-\epsilon, \epsilon) \to M \\ &c(0) = p \end{aligned}$$

where $[c'(0)]f = \frac{d}{dt}(f \circ c)\big|_{t=0}$.Note that the curve c need not be unique. In local coordinates we have that

$$\begin{aligned} c'(0)f &= \frac{d}{dt}(f \circ c)\Big|_{t=0} \\ &= \frac{d}{dt}(\tilde{f} \circ \tilde{c})\Big|_{t=0} \\ &= \frac{\partial \tilde{f}}{\partial x_i}(\varphi(p)) \cdot \underbrace{\frac{d}{dt}c_i\Big|_{t=0}}_{=a_i} \\ &= a_i \frac{\partial \tilde{f}}{\partial x_i}\Big|_{\varphi(p)} \end{aligned}$$

Definition 1.10. We define the **tangent space** of a smooth manifold M at a point $p \in M$ as

$$T_pM := \{X_c \text{ vector at } p\} \tag{2}$$

With the notion of the tangent space, we can return to smooth functions between manifolds. As before,

Remark. Intuitively, we understand the tangent spaces T_pM and $T_{\varphi(p)}N$ to be linear approximations of their respective manifolds, and then Df_p is the linear approximation of the smooth function f.

Theorem 1.11. *Let M and N be smooth manifolds and $f : M \to N$ be smooth. If f is a diffeomorphism, then Df_p is an isomorphism for any $p \in M$.*

Remark. The converse of the above statement is presented as follows: If Df_p is an isomorphism, then f is a **local diffeomorphism**, meaning that there exists a neighborhood U of p and V a neighborhood of $f(p)$ such that $f\big|_U : U \to V$ is a diffeomorphism.

Definition 1.12. Let M and N be smooth manifolds of orders m and n, respectively. If Df_p is injective for all $p \in M$, then we say that f is an **immersion**. Moreover, if $f : M \to f(M) \subset N$ is a homeomorphism, then we say that f is an **embedding** and $f(M)$ is a **submanifold**.

Proposition 1.13. *If Df_p is injective, then there exist chars (U, φ) and (V, ψ) at p and $f(p)$, respectively, such that $\tilde{f} = \psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$*

We now consider some examples of the above definitions.

Example 1.14. 1.

Remark. If f is an immersion, then f is locally an embedding.

1.4 Tangent Bundles and Vector Fields

1.4.1 Tangent Bundle

Definition 1.15. Let M be a smooth manifold. We define the **tangent bundle** as the following

$$TM = \bigcup_{p \in M} T_pM. \tag{3}$$

Let M be a smooth manifold with an atlas $\mathcal{U}$. Consider a chart (U, φ). It follows that

$$\begin{array}{ccc} TU = \bigcup_{p \in U} T_pM & \xrightarrow{\hat{\varphi}} & \varphi(U) \times \mathbb{R}^n \\ \downarrow & & \\ = \{p, v_p \mid p \in U, v_p \in T_pM\} & & \end{array}$$

Theorem 1.16. *Let M be a smooth manifold and consider the tangent bundle TM. Then, the following are true.*

1. *TM is a manifold with a canonical smooth structure.*
2. *The map $\pi : TM \to M$ given by $(p, X) \mapsto p$ is a submersion.*
3. *If $p \in M$, then $\pi^{-1}(p) = T_pM$.*

1.4.2 Vector Fields

Definition 1.17. Let M smooth manifold. A **vector field** is a smooth mapping $X : M \to TM$ such that

$$\pi \circ X = id_M.$$

We call X a **section** of the tangent bundle given by

$$\Gamma(TM) \coloneqq \{C^\infty(M) \ni X : M \to TM \mid \pi \circ X = id_M\}$$

Definition 1.18. Let M be a smooth manifold and $X, Y \in \Gamma(TM)$. We define the **Lie bracket** of X and Y as

$$[X, Y] = XY - YX. \tag{4}$$

Proposition 1.19. *Let X, Y, Z be vector fields on a smooth manifold M. Then the following hold:*

1. *$[X, Y]$ is a vector field on M, called the **Lie bracket** of X and Y.*
2. *$[X, Y] = -[Y, X]$*
3. *$[aX + bY, Z] = a[X, Y] + b[Y, Z]$*
4. *$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$*
5. *$[fX, gY] = fg[X, Y] + fX(g) \cdot Y - gY(f) \cdot X$.*

1.5 Tensors on a Riemannian Manifold

Let M be a smooth manifold of dimension n. For each point $p \in M$, denote by T_pM the tangent space at p and by T_p^*M its dual, the cotangent space. A *tensor of type* (k, ℓ) at p is a multilinear map

$$T : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{k \text{ times}} \times \underbrace{T_pM \times \cdots \times T_pM}_{\ell \text{ times}} \to \mathbb{R}.$$

The space of such tensors is denoted by

$$\mathcal{T}^k_\ell(T_pM) := \underbrace{T_pM \otimes \cdots \otimes T_pM}_{k \text{ times}} \otimes \underbrace{T_p^*M \otimes \cdots \otimes T_p^*M}_{\ell \text{ times}}.$$

A *tensor field* of type (k, ℓ) on M assigns to each point $p \in M$ a tensor in $\mathcal{T}^k_\ell(T_pM)$ in a smooth manner. The collection of all smooth (k, ℓ)-tensor fields is denoted by $\mathcal{T}^k_\ell(M)$.

In local coordinates $(x^1, \ldots, x^n)$ on a chart $U \subset M$, a tensor field $T \in \mathcal{T}^k_\ell(M)$ can be expressed as

$$T = T^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell} \frac{\partial}{\partial x^{i_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{i_k}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_\ell},$$

where the coefficients $T^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell}$ are smooth functions on U.

Under a change of coordinates, these components transform according to the tensor transformation law:

$$\widetilde{T}^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell} = \frac{\partial \widetilde{x}^{i_1}}{\partial x^{a_1}} \cdots \frac{\partial \widetilde{x}^{i_k}}{\partial x^{a_k}} \frac{\partial x^{b_1}}{\partial \widetilde{x}^{j_1}} \cdots \frac{\partial x^{b_\ell}}{\partial \widetilde{x}^{j_\ell}} T^{a_1 \cdots a_k}{}_{b_1 \cdots b_\ell}.$$

A Riemannian metric g on M is a smooth, symmetric $(0, 2)$-tensor field such that for each $p \in M$, the bilinear form $g_p : T_pM \times T_pM \to \mathbb{R}$ is positive definite. In local coordinates, it can be written as

$$g = g_{ij}\, dx^i \otimes dx^j,$$

where (g_{ij}) is a symmetric, positive-definite matrix at each point.

The metric induces an isomorphism between the tangent and cotangent bundles. Specifically, for each vector field $X \in \mathfrak{X}(M)$, one defines the associated 1-form $X^\flat$ by

$$X^\flat(Y) := g(X, Y), \quad \forall Y \in \mathfrak{X}(M).$$

This operation is called the *musical isomorphism*. Its inverse is denoted by $\sharp$.

Using the metric, one can raise and lower indices of tensors. For instance, given a $(0, 2)$-tensor T_{ij}, one obtains a $(1, 1)$-tensor by

$$T^i{}_j = g^{ik} T_{kj},$$

where (g^{ij}) is the inverse matrix of (g_{ij}).

Tensors play a central role in Riemannian geometry, as geometric quantities such as curvature, connections, and differential operators are naturally expressed in tensorial form.

2 Riemannian Manifolds

We now begin to study Riemannian manifolds. To motivate this, we begin with the question of measuring distance on a manifold. We recall from calculus that this process requires us to define the length of a vector.

2.1 Riemannian Metrics

Let M be a smooth manifold. Consider an inner-product on T_pM denoted by $\langle \cdot, \cdot \rangle_p$ which is symmetric, bilinear, and positive definite.

Definition 2.1. A **Riemannian metric** on a smooth manifold M is a correspondence that assigns to each point $p \in M$ an inner product given by $\langle \cdot, \cdot \rangle_p$ in the tangent space T_pM that varies smoothly on p.

Definition 2.2. A smooth manifold M coupled with a Riemannian metric g is called a **Riemannian manifold**, and we denote it as (M, g).

Example 2.3. We consider the basic case of $M = \mathbb{R}^n$, called the **Euclidean case**. Let $x = (x_1, ..., x_n)$ and $u, v \in T_x\mathbb{R}^n \cong \mathbb{R}^n$. Then, we can define the Euclidean metric as the following:

$$\langle u, v \rangle_{Euc} := u_i v_i$$

where repeated indices indicate summation.

Definition 2.4. We define a **partition of unity** for a smooth manifold M. Given a covering $(U_\alpha, \varphi_\alpha)$ of M, it admits a countable subcovering $(V_i, \varphi_i)_{i \in I}$ and $f_i : M \to \mathbb{R}$ such that

1. $\text{supp}(f_i) \subset V_i \subset U_\alpha$ for some α.

2. For any $p \in M$, there exists a neighborhood W of p such that W intersects only finitely many V_i.

3. $\sum_{i\in I} f_i(p) = 1$.

Theorem 2.5. *Any smooth manifold with a countable basis admits a Riemannian metric.*

Assumption. For the rest of these notes, we assume that any smooth manifold M has a countable basis. We make this assumption because a countable basis yields a partition of unity on M.

Proof. We provide a sketch of the proof. Let $(U_\alpha, \varphi_\alpha)$ be a covering of M by local charts. As M admits a countable basis, there exists a finite subcovering (V_i, φ_i) and $\{f_i\}$ a subordinate partition of unity. Define on the local chart V_i an inner-product in the following way: Let $q \in V_i$ and $u, v \in T_qV_i$. Define

$$\langle u, v\rangle_{i,q} := \langle (D\varphi_i)_q(u), (D\varphi_i)_q(v)\rangle_{Euc}.$$

We now define an inner-product on the entire manifold M as follows: for any $p \in M$ and $u, v \in T_pM$, define

$$\langle u, v\rangle_p := \sum_{i\in I} f_i(p)\langle u, v\rangle_{i,p}.$$

As $\operatorname{supp}(f_i) \subset V_i$, it follows that $f_i\langle \cdot, \cdot\rangle_i$ is symmetric, bilinear and smooth on M. This immediately yields that $\langle \cdot, \cdot\rangle_p$ is symmetric, bilinear, and smooth and M. We need only show that $\langle \cdot, \cdot\rangle_p$ is positive definite. Suppose that $p \in M$ and $u \in T_pM$ and consider $\langle u, u\rangle_p = \sum_{i\in I} f_i(p)\langle u, u\rangle_{i,p} \geq 0$. Suppose that $\langle u, u\rangle_p = 0$ if and only if $\langle u, u\rangle_{i,p} = 0$ if and only if $u = 0$. Hence, we have constructed the Riemannian metric. □

Definition 2.6. Let (M, g_M) and (N, g_N) be Riemannian manifolds and $f : M \to N$ be a diffeomorphism. We call f an **isometry** if

$$g_M(u, v) = g_N((Df)_p(u), (Df)_p(v))$$

for all $p \in M$. We call f a **local isometry** if, for all $p \in M$, there exists a neighborhood U of p in M such that $f\big|_U : U \to f(U) \subset N$ is an isometry.

We next present an example of how to induce Riemannian metrics given special types of smooth maps.

Example 2.7. Let M and N be smooth manifolds and $f : M \to N$ be an immersion. If (N, g_N) is a Riemannian manifold, then M has an induced Riemannian metric given by the following: for all $p \in M$ and $u, v \in T_pM$ define

$$g_M(u, v)_p := g_N((Df)_p(u), (Df)_p(v))_{f(p)}.$$

We now consider some important examples:

2.1.1 Important Examples

1. Consider the n-dimensional unit sphere S^n. If there exists an immersion $i : S^n \hookrightarrow \mathbb{R}^n$ where $\mathbb{R}^n$ is a Riemannian manifold with the Euclidean metric. Then, there exists an induced metric on S^n called the canonical metric.

2. Let H be a group that acts properly discontinuously on the manifold (M, g_M) such that for any $h \in H$, $\varphi_h : M \to M$ is an isometry. Then, M/H has an induced Riemannian metric. To see this, consider $\hat{p} \in M/H$ and $\hat{u}, \hat{v} \in T_{\hat{p}}(M/H)$, and let $p \in \pi^{-1}(\hat{p})$ and U be a neighborhood of p such that $\Pi_U : U \to \pi(U) \subset M/H$ is a diffeomorphism. Define

$$g(\hat{u}, \hat{v})_{\hat{p}} \coloneqq g_M((D\Pi_U)^{-1}(\hat{u}), (D\Pi_U)^{-1}(\hat{v}))_p$$

3. **The Product Metric**. Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds. From these two manifolds, we can construct a new product manifold: $M_1 \times M_2 = \tilde{M}$. This construction has the induced product metric: $\tilde{g} = g_1 \times g_2$. We define the product metric in the following way: let $(p, q) \in \tilde{M}$. We have

$$T_{(p,q)} M_1 \times M_2 = T_p M_1 \oplus T_q M_2$$

Then, for $u \in T_{(p,q)} M_1 \times M_2$

$$u \mapsto \left((D\pi_1)_{(p,q)}\, u, (D\pi_2)_{(p,q)}\, u \right)$$

where π_i is the projection on the i^{th} component. If $(u, v) \in T_{(p,q)} M_1 \times M_2$, then we define the metric as

$$\tilde{g}(u, v) \coloneqq g_1 \left((D\pi_1)_{(p,q)}\, u, (D\pi_1)_{(p,q)}\, v \right) + g_2 \left((D\pi_2)_{(p,q)}\, u, (D\pi_2)_{(p,q)}\, v \right)$$

It is left as an exercise to show that $\tilde{g}$ is bilinear, symmetric, and positive definite.

4. **Flat Tori**. Let Γ be a lattice in $\mathbb{R}^n$ generated by a basis of the form $\{\vec{a_1}, ..., \vec{a_n}\}$. This is the group acting by translations on $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$. This action acts properly discontinuously by isometries. Then, $\mathbb{R}^n/\Gamma$ has the induced quotient metric and $(R^n/\Gamma, g_{\text{quot}})$ is called a **flat torus**. It is the case that $\mathbb{R}^n/\Gamma$ is diffeomorphic to the torus T^n.

Proposition 2.8. *Let Γ_1 and Γ_2 be two lattices generated by bases $\{\vec{a_1}, ..., \vec{a_n}\}$ and $\{\vec{b_1}, ..., \vec{b_n}\}$. Then, $(\mathbb{R}^n/\Gamma_1, g_1)$ and $(\mathbb{R}^n/\Gamma_2, g_2)$ are isomorphic if and only if there exists an isometry of $\mathbb{R}^n$ which maps Γ_1 into Γ_2.*

Definition 2.9. Let M be a Riemannian manifold and g and h be Riemannian metrics. We say that g and h are **homothetic** if there exists $\lambda > 0$ such that $g = \lambda h$.

We now consider the case of $n = 2$ for the flat torus and discuss how flat tori are classified.

Theorem 2.10. *The flat tori* $(\mathbb{R}^n/\Gamma, \langle\cdot,\cdot\rangle_{quot})$ *are classified up to homotheties and isometries by the region:*

$$\mathcal{M} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;\middle|\; x^2 + y^2 \geq 1 \;,\; 0 \leq x \leq 1/2 \;,\; y > 0 \right\}$$

where Γ*– up to homothety and isomorphism– is generated by* $\vec{a_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *and* $\vec{a_2} = \begin{pmatrix} x \\ y \end{pmatrix}$*.*

Proof. We provide a sketch of the proof. Consider the flat torus $\left(\mathbb{R}^2/\Gamma, \langle\cdot,\cdot\rangle_{\text{quot}}\right)$ and let $\vec{a_1}$ be *one* of the shortest vectors of Γ. We emphasize *one* to include the case of negative $-\vec{a_1}$. After rescaling (which is a homothety $\frac{1}{\lambda} = |\vec{a_1}|$). We can assume that $\vec{a_1}$ has uit length. After a rotation (an isometry), we can assume that $\vec{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Let $\vec{a_2} \in \Gamma \backslash \{na_1 \mid \vec{n} \in \mathbb{Z}\}$, one of the shortest vectors in the set. Then, $|\vec{a_2}| \geq |\vec{a_1}| = 1$. It follows that after considering reflections on the x-axis and y-axis and possibly switching $\vec{a_1}$ by $-\vec{a_1}$, we can assume that $\vec{a_2}$ lies in the first quadrant. Next, consider the quantity $\vec{a_2} - \vec{a_1} = \begin{pmatrix} x-1 \\ y \end{pmatrix}$ such that $\vec{a_2} = \begin{pmatrix} x \\ y \end{pmatrix}$. If $x \geq 1$, $\vec{a_2}' = \vec{a_2} - \vec{a_1}$ is shorter, which yields a contradiction on the assumption that $\vec{a_2}$ is of the shortest length. This is given by

$$(x-1)^2 + y^2 < x^2 + y^2.$$

If $x > 1/2$, we can use the same argument as above to arrive at a contradiction. We already showed above that any flat torus is equivalent (up to homotety and isometry) to a torus $\mathbb{R}^2/\Gamma$ where Γ is the lattice generated by $\vec{a_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{a_2}$. Hence, it is enough to show that two distinct vectors $\vec{a_1}, \vec{b} \in M$, give a non-isometric flat tori $\mathbb{R}/\Gamma$ where Γ is a lattice generated by $\vec{a_1}$ and $\vec{a_2}$ and $\mathbb{R}^2/\Gamma'$ where Γ' is a lattice generated by $\vec{a_1}$ and $\vec{b}$. Proposition (**??**) yields that $\left(\mathbb{R}^2/\Gamma, \langle\cdot,\cdot\rangle_{\text{quot}}\right)$ is isometric to $\left(\mathbb{R}^2/\Gamma', \langle\cdot,\cdot\rangle_{\text{quot}}\right)$ if and only if there exists a function $F : \mathbb{R}^2 \to \mathbb{R}^2$ an isometry such that $F(\Gamma) = \Gamma'$ and F is linear. If □

2.2 Metric Space Structure

2.2.1 Length of Curves

Definition 2.11. Let (M, g) be a Riemannian manifold and consider a curve $c : I \to M$ that is smooth. For $[a, b] \subset I$, we define the **length of** c **from** a **to** b as

$$\ell_a^b(c) := \int_a^b \left(g_{c(t)}\left(c'(t), c'(t)\right)\right)^{1/2} \, dt \tag{5}$$

Proposition 2.12. *Let* (M_1, g_1) *and* (M_2, g_2) *be two Riemannian manifolds and let* $F : (M_2, g) \to (M_2, g_2)$ *be an isometry (a local isometry will be enough for the forthcoming result). Then for any smooth curve* $c : I \to M$ *and* $[a, b] \subset I$*, we have*

$$\ell_a^b(c) = \ell_a^b(f \circ c).$$

Proof. We follow our nose in the following argument:

$$\begin{aligned}
\ell_a^b\left(f\circ c\right) &= \int_a^b \left(g_{f\circ c}\left(\frac{d}{dt}\left(f\circ c\right), \frac{d}{dt}\left(f\circ c\right)\right)\right)^{1/2}\ dt\\
&= \int_a^b \left(g_{f\circ c}\left(Df(c'(t)), Df(c'(t)\right)\right)^{1/2}\ dt\\
&= \int_a^b g_{c(t)}\left(c'(t), c'(t)\right)^{1/2} dt
\end{aligned}$$

where the last inequality follows from the isometry. □

Corollary 2.13. *Let $F : (\mathbb{R}^n, \langle\cdot,\cdot\rangle) \to (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$ be an isometry between $\mathbb{R}^n$ with the Euclidean inner product. Then, F preserves the Euclidean distance, F is linear, and the matrix representation A_F is orthonormal.*

Definition 2.14. Let (M, g) be a Riemannian manifold and consider a curve $c : I \to M$ that is smooth. For $[a, b] \subset I$, we define the **Energy from** a **to** b as

$$E_a^b(c) \coloneqq \int_a^b g_{c(t)}\left(c'(t), c'(t)\right)\ dt \tag{6}$$

2.2.2 Volume

In Euclidean geometry, we define volume as follows: given an orthonormal basis $\{e_1, ..., e_n\}$, the volume of the unit cube is one:

$$\text{vol(unit cube)} = 1.$$

If we consider another basis $\{u_1, ..., u_n\}$ for $\mathbb{R}^n$ that need not be orthonormal, then the basis vectors form a parallelepiped. We recall that the volume of a parallelepiped as

$$\text{vol(parallelepiped)} = \det\left(u_1, ..., u_n\right).$$

If we move the setting to a Riemannian manifold (M, g), consider $p \in M$ and a chart (U, φ). Let $\left\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right\}$ be a basis for the tangent space $T_{\varphi(p)}\mathbb{R}^n$ and let $(x_1, ..., x_n)$ be a basis for the tangent space T_pM where

$$x_i = D\varphi^{-1}_{\varphi(p)}\left(\frac{\partial}{\partial x_i}\right).$$

The coefficients of the metric $g_{ij}(p) = g_p(x_i, x_j)$. Let $\{e_1, ..., e_n\}$ be an orthogonal basis compatible with the orientation of $\{x_1, ..., x_n\}$ for (T_pM, g). Hence, we can right $x_i = a_{i,k}e_k$ which yields

$$\begin{aligned}
g_{ij} &= g(x_i, x_j)\\
&= g(a_{i,k}e_k, a_{j,\ell}e_\ell)\\
&= \sum_{k,\ell} a_{i,k}\cdot a_{j,\ell}\ g(e_k, e_\ell)\\
&= \sum_k a_{i,k}\cdot a_{j,k}
\end{aligned}$$

This means that

$$(g_{ij})_{(i,j)} = (a_{i,k})_{(i,k)} \cdot (a_{j,k})^T.$$

We therefore find that the volume of a parallelepiped with side $x_1, ..., x_n$ in T_pM is given by

$$\det((a_{i,k})_{i,k}).$$

However, this formula is cumbersome and difficult to calculate. We then turn to the above calculation, connecting to the metric. It follows that

$$\mathrm{vol}(\text{parallelepiped}) = \sqrt{\det(g_{ij})}$$

Definition 2.15. Let (M, g) be a Riemannian metric and consider a local chart (U, φ) and a compact solid R such that $R \subset U \subset M$. Then we define the volume of R as

$$\mathrm{vol}(R) = \int_{\varphi(R)} 1 \cdot \sqrt{\det(g_{ij})}\, dx_1 \cdots dx_n. \tag{7}$$

Remark. The above definition is well-defined and independent of the chart.

Definition 2.16. Let (M, g) be an oriented Rienannian manifold. We define μ_g as an n-dimensional nowhere zero n-form such that

$$\mu_g|_U = \sqrt{\det(g_{ij})} dx_1 \wedge \cdots \wedge dx_n. \tag{8}$$

We call μ_g the **Riemannian volume form**.

Definition 2.17. Let (M, g) be a Riemannian manifold and $\overline{R} \subset M$ be a compact solid. Let $(U_\alpha, \varphi_\alpha)$ be a finite covering of $\overline{R}$ of local charts and $\{f_\alpha\}$ be a subordinate partition of unity. Define the volume of $\overline{R}$ as

$$\mathrm{vol}(R) := \sum_\alpha \int_{\varphi_\alpha(U_\alpha \cap \overline{R})} f_\alpha\left(\varphi^{-1}(x)\right) \sqrt{\det(g_{ij})}\, dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \tag{9}$$

Proposition 2.18. *Definition (9) is independent of charts and partition of unity.*

Proof. We provide a sketch of the proof. Let (V, ψ) be another chart and (h_{ij}) be a matrix representation of g in the chart. For $\overline{R} \cap V$:

$$\begin{aligned}
&\sum_\alpha \int_{(U \cap V \cap \overline{R})} f_\alpha(\varphi_\alpha^{-1}(x^\alpha)) \sqrt{\det(g_{ij}^\alpha)}\, dx_1^\alpha \wedge \cdots \wedge dx_n^\alpha \\
&= \sum_\alpha \int_{\psi(U \cap V \cap \overline{R})} f_\alpha(\psi^{-1}(y)) \sqrt{\det(h_{ij})}\, dy_1 \wedge \cdots \wedge dy_n \\
&= \int_{\psi(V \cap \overline{R})} \sum_\alpha f_\alpha\left(\psi^{-1}(y)\right) \sqrt{\det(h_{ij})}\, dy_1 \wedge \cdots \wedge dy_n \\
&= \int_{\psi(V \cap \overline{R})} \sqrt{\det(h_{ij})}\, dy_1 \wedge \cdots \wedge dy_n = \mathrm{vol}(V \cap \overline{R}).
\end{aligned}$$

□

Remark. We can think of the metric g in the following way. If $X, Y \in \Gamma(M)$ and $p \in M$, we have $g(X,Y)_p = g_p(X_p, Y_p) \in \mathbb{R}$. We have that g is a (0,2) tensor. In other words, $g \in \Gamma(T^*M \otimes T^*M)$. We adopt the following notation for the metric on a chart U:

$$g|_U = \sum g_{ij}\ dx_i \otimes dx_j$$

2.2.3 Riemannian Submersions

Let (M,g) and (N,h) be Riemannian manifolds of degree m and n, respectively. Let $f : M \to N$ be a submersion. We have that $\ker(Df_p) \subset T_pM$, which is a $m-n$ dimensional subspace. Define the space

$$H_p := \ker(Df_p)^{\perp} = \{v \in T_pM \mid g(u,v) = 0 \text{ for all } u \in \ker(Df_p)\}.$$

It follows that $Df_p|_{H_p} : H_p \to T_{f(p)}N$ is an isomorphism.

Definition 2.19. We say f is a **Riemannian submersion** if Df_p preserves the inner-product $g_p|_{H_p}$ on H_p and h on $T_{f(p)}N$.

Example 2.20. The map

3 Affine Connections

We begin our discussion of affine connections by quickly reviewing how Lie derivative of vector fields behave. Our notion of an affine connection will arise from a desire to rid ourselves of certain inconveniences that Lie derivatives hold.

3.1 Lie Derivatives

We have already covered how to make sense of directional derivatives of real-valued functions on a manifold. Indeed, a tangent vector $v \in T_pM$ is by definition an operator that acts on a smooth function f to give a number vf that we interpret as a directional derivative of f at p.

What about the directional derivative of a vector field? Let's first consider how this would play out in Euclidean space. It makes sense to define the directional derivative of a smooth vector field X in the direction of a vector $v \in T_p\mathbb{R}^n$. It is the vector

$$D_vX(p) = \frac{d}{dt}\bigg|_{t=0} X_{t+pv} = \lim_{t\to 0} \frac{X_{p+tv} - X_p}{t}.$$

We can easily calculate the directional derivative by applying D_v to each component of W separately:

$$D_vW(p) = D_vW^i(p)\frac{\partial}{\partial x^i}\bigg|_p.$$

This definition is hard to generalize, however. The reason is that we are implicitly using the fact that $\mathbb{R}^n$ is a vector space. That is, the tangent vectors W_{p+tv} and W_p can both be viewed as elements of $\mathbb{R}^n$.

3.1.1 Lie Derivatives on Vector Fields

Suppose we try to generalize this a manifold M. To begin, we make the replacement of $p + tv$ by a curve $\gamma(t)$ that starts at the point p and whose initial velocity is v. However, this substitution still yields a fundamental error; the vectors $W_{\gamma(t)}$ and $W_{\gamma(0)}$ belong to two different spaces: $T_{\gamma(t)}M$ and $T_{\gamma(0)}M$, respectively. This was negated in the case of $\mathbb{R}^n$ because there is a canonical identification of each tangent space with $\mathbb{R}^n$ itself; but, on a generic manifold there is no such identification. Thus, there is no coordinate independent way

We fix this problem if we replace the vector $v \in T_pM$ with a vector field $X \in$, so we can use the flow of X to push back values of W back to p and then differentiate. We can now make the following definition.

Definition 3.1. Suppose that M is a smooth manifold, X is a smooth vector field on M, and F is the flow of X. For any smooth vector field Y on M, define a rough vector field on M, denoted by the $\mathscr{L}_X Y$ and call the **Lie derivative of Y with respect to X**, by

$$\begin{aligned}(\mathscr{L}_X W)_p &= \frac{d}{dt}\bigg|_p d(F_{-t})_{F_t(p)}(Y_{F_t(p)}) \\ &= \lim_{t\to 0} \frac{d(F_{-t})_{F_t(p)}(Y_{F_t(p)}) - Y_p}{t},\end{aligned}$$

provided the derivative exists. For small $t \neq 0$, at least the difference quotient makes sense: F_t is defined in a neighborhood of p, and F_{-t} is the inverse of F_t, so the objects $d(F_{-t})_{F_t(p)}(Y_{F_t(p)})$ and Y_p are elements of the tangent space T_pM.

Example 3.2. Let $M = \mathbb{R}^2$ and consider the vector fields

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad\qquad Y = \frac{\partial}{\partial x}.$$

Let $p = (x, y)$ and lets calculate the Lie derivative $\mathscr{L}_X Y$. Using the above definition, we need to calculate the flow of X. It follows that we have the system

$$\begin{aligned}\frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \\ (x(0), y(0)) &= (x, y).\end{aligned}$$

We can hence write an equation for $x(t)$ as

$$x''(t) + x(t) = 0$$

which has the solution of

$$x(t) = x\cos(t) + y\sin(t).$$

We get that the flow $F_t(x, y)$ is given by

$$F_t(x,y) = \begin{pmatrix}\cos(t) & -\sin(t) \\ \sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$$

We have that $Y_{F_t(p)} = \dfrac{\partial}{\partial x}|_{F_t(p)}$

The above definition can be rather tedious. We present an alternate way to calculate Lie derivatives. First, we need the following lemma on the aforementioned Lie brackets.

Lemma 3.3. *Let X, Y be smooth vector fields on a manifold M with or without boundary,and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ in terms of the smooth local coordinates (x^i) for M. Then*

$$[X, Y] = (X^i \frac{\partial Y^j}{\partial x^i} - X^j \frac{\partial x^j}{\partial x^i}) \frac{\partial}{\partial x^j} = (XY^j - YX^j) \frac{\partial}{\partial x^j}.$$

Proof. Because $[X, Y]$ is a smooth vector field, it suffices to check this on a smooth chart. We have

$$\begin{aligned} [X, Y]f &= X^i \frac{\partial}{\partial x^i}(Y^j \frac{\partial f}{\partial x^j}) - Y^j \frac{\partial}{\partial x^j}(X^i \frac{\partial f}{\partial x^i}) \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} - Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\ &= X^i \frac{\partial Y^j}{\partial x^i} \frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j} \frac{\partial f}{\partial x^i} \end{aligned}$$

□

Lemma 3.4. *Let V be a smooth vector-field on a smooth manifold M, and let $p \in M$ be a regular point of V. There exists smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\frac{\partial}{\partial s^1}$.*

Theorem 3.5. *If M is a smooth manifold and X_1, X_2 are vector fields, then $\mathscr{L}_{X_1} X_2 = [X_1, X_2]$.*

Proof. Suppose that V and W are vector fields of a smooth manifold M. Define the set $R(X_1)$ as the set of points $p \in M$ such that $p \in M$ such that $X_1(p) \neq 0$. Note that $R(V)$ is open in M by continuity, and its closure is the support of V. We want to show that

$$(\mathscr{L}_V W)_p = [V, W]_p$$

for all $p \in M$ by considering the following three cases.
Case 1: $p \in R(V)$. In this case, we can choose smooth coordinates (u^i) on a neighborhood of p in which V has the coordinate representation $V = \frac{\partial}{\partial u^1}$ by the lemma. Therefore, the denote the flow of V by $F_t(u) = (u^1 + t, u^2, \ldots, u^n)$. Since F_{-t} is just a translation, $d(F_{-t})_{F_t(x)}$ is just the identity at every point $x \in M$. Thus, for any $u \in U$,

$$\begin{aligned} d(F_{-t})_{F_t(u)}(W_{F_t(u)}) &= d(F_{-t})_{F_t(x)}(W^j(u^1 + t, \ldots, u^n) \frac{\partial}{\partial u^j}\bigg|_{F_t(u)}) \\ &= W^j(u^1 + t, \ldots, u^n) \frac{\partial}{\partial u^j}\bigg|_u. \end{aligned}$$

Therefore, by the definition of Lie derivative, we have

$$(\mathscr{L}_V W)_u = \frac{\partial}{\partial t}\bigg|_{t=0} W^j(u^1+t,\ldots,u^n)\frac{\partial}{\partial u^j}\bigg|_u = \frac{\partial W^j}{\partial u^1}(u^1,\ldots,u^n)\frac{\partial}{\partial u^j}\bigg|_u.$$

However, using the lemma we see that

$$[V,W]\bigg|_u = \sum_{j=1}^{n}(V(W^j)-W(V^j))\frac{\partial}{\partial u^j}\bigg|_u = \sum_{j=1}^{n}\frac{\partial W^j}{\partial u^1}(u^1,\ldots,u^n)\frac{\partial}{\partial u^j}\bigg|_u$$

Since $V = \frac{\partial}{\partial u^1}$.

Case 2: Let $p \in \operatorname{supp}(V)$, Because the $\operatorname{supp}(V)$ is the closure of $R(V)$, there is a sequence (p_i) that converges to p. By case one, we know that $(\mathscr{L}_V W)_{p_i} = [V,W]_{p_i}$ for every term in the sequence. Thus,

$$(\mathscr{L}_V W)_p = \lim_{i\to\infty}(\mathscr{L}_V W)_{p_i} = \lim_{i\to\infty}[V,W]_{p_i} = [V,W]_p$$

Case 3: $p \in M - \operatorname{supp}(V)$. In this case, $V = 0$ in a neighborhood of p. On one hand, this implies that the flow is equal to the identity map in a neighborhood of p for all t. So,

$$(\mathscr{L}_X W)_p = \frac{d}{dt}\bigg|_p d(F_{-t})_{F_t(p)}(W_{F_t(p)}) = \frac{d}{dt}\bigg|_p W_p = 0$$

since W_p does not depend on t. Also, $[V,W]\big|_p$ is also zero since $V(p) = 0$. □

This theorem provides us with a geometric interpretation of the Lie bracket of two vector fields: it is the directional derivative of the second vector field along the flow of the first. We now present the following properties of the Lie derivative.

Corollary 3.6. *Suppose that M is a smooth manifold and V, W, X are smooth vector fields on M. It follows that*

1. $\mathscr{L}_V W = -\mathscr{L}_W V$
2. $\mathscr{L}_L[V,X] = [\mathscr{L}_V W, X] + [W, \mathscr{L}_V X]$
3. $\mathscr{L}_{[V,W]}X = \mathscr{L}_V\mathscr{L}_W X - \mathscr{L}_W\mathscr{L}_V X$
4. *If $f \in C^\infty(M)$, then the following two properties hold:*

$$\mathscr{L}_V(fW) = (Vf)\cdot W + f\cdot\mathscr{L}_V W$$
$$\mathscr{L}_{fV}W = f\cdot\mathscr{L}_V W - W(f)\cdot V$$

3.2 Affine Connections

The reason for introducing affine sections arises from the desire to rid ourselves of the red term in the above corollary. Instead, we would like a derivative that obeys

$$D_{fV}W = f \cdot D_V W.$$

This motivates the following definition.

Definition 3.7. Let M be a differentiable manifold and consider

$$\nabla : \Gamma(M) \times \Gamma(M) \to \Gamma(M),$$

a $C^\infty(M \times M)$ mapping that satisfies the following

(i) $\nabla_{(fX+gY)}Z = f\nabla_X Z + g\nabla_Y Z$ for all $f, g \in C^\infty(M)$ and $X, Y, Z \in \Gamma(M)$

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$ for all $X, Y, Z \in \Gamma(M)$

(iii) $\nabla_X(f \cdot Y) = X(f)Y + f \cdot \nabla_X Y$.

We call ∇ the **affine connection** and the object $\nabla_X Y$ is called the **covariant derivative** of Y with respect to X.

Remark. 1. If ∇_1 and ∇_2 are two affine connections, then $\nabla_1 + \nabla_2$ is *not* an affine connection. Indeed, condition (iii) in the above definition yields a bad coefficient:

$$(\nabla_1 + \nabla_2)_X(f \cdot Y) = 2X(f) \cdot Y + f(\nabla_1)_X Y + f(\nabla_2)_X Y.$$

To mitigate this annoyance in the definition, we note that

$$\alpha\nabla_1 + (1 - \alpha)\nabla_2$$

is an affine connection for any $\alpha \in \mathbb{R}$.

2. If ∇ is an affine connection and $L : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$ is $C^\infty(M)$ bilinear, then $\nabla + L$ is an affine connection.

We now discuss affine connections in local charts. Let M be a smooth manifold and (U, φ) be a local chart. We have that $x_i = (D\varphi^{-1})\dfrac{\partial}{\partial x_i}$ is a basis for T_pM for $p \in U$. Then, if $X, Y \in \Gamma(M)$, we have $X|_U = a_i x_i$ and $Y|_U = b_i x_i$ where $a_i, b_i \in C^\infty(U)$. It follows that

$$\begin{aligned}\nabla_X Y &= \sum_i a_i \nabla_{x_i} Y \\ &= \sum_{i,j} a_i x_i(b_j)x_j + a_i b_j \nabla_{x_i} y_j\end{aligned}$$

Hence, we have to calculate the covariant derivative of basis elements to calculate the covariant derivatives of vector fields. We have that

$$\nabla_{x_i} y_i = \sum_k \Gamma_{ij}^k x_k. \tag{10}$$

The coefficients of the resulting vector field are very important.

Definition 3.8. We call the coefficients Γ_{ij}^k as **Christoffel symbols** and we can write the covariant derivative with two vector fields as

$$\nabla_X Y = \sum_k \left(\sum_i a_i x_i b_k + \sum_{i,j} a_i b_j \Gamma_{ij}^k \right) x_k \tag{11}$$

Proposition 3.9. *Equation (11) defines an affine connection and* Γ_{ij}^k.

Remark. 1. Affine connections determines and is determined by the Christoffel symbols.

2. $(\nabla_X Y)_p$ is computed in terms of the coefficients of the vector fields X and Y and the first-order derivatives of the coefficients of the vector field Y.

3.2.1 Derivatives of vector fields along curves

Let $I \subset \mathbb{R}$ and M be a smooth manifold. Let I be an open interval, and let $c : I \to M$ be a smooth curve. The tangent vector is given by $Dc_t\left(\frac{\partial}{\partial t}\right) := c'(t)$. Consider an arbitrary vector field Y defined on M and a vector field V defined along the curve c. V is defined by $V : I \to TM$ and $V(t) \in T_{c(t)}M$. The following diagram is also commutative:

$$\begin{array}{ccc} V : I & \xrightarrow{V} & TM \\ & \searrow^{c} & \downarrow \pi \\ & & M \end{array}$$

Proposition 3.10. *Let M be a differentiable manifold and ∇ be an affine connection. Then there exists a unique correspondence that associates each vector field V along a curve $c : I \to M$ with another vector field $\frac{Dv}{dt}$ along c such that*

(i) $\frac{D}{dt}(V + W) = \frac{D}{dt}(V) + \frac{D}{dt}(W)$

(ii) $\frac{D}{dt}(fV) = \frac{df}{dt} \cdot V + f\frac{D}{dt}(V)$

(iii) If V is the restriction of $Y \in \Gamma(M)$, then $\frac{D}{dt}(V) = \nabla_{c'(t)}Y$.

Proof. It is enough to prove the result in a local chart. Let (U, φ) be a local chart and assume that $c(I) \subset U \subset M$. We have $x_i = (D\varphi^{-1})\left(\frac{\partial}{\partial x_i}\right)$ as a basis.

Uniqueness: Let V be a vector field whose decomposition is $V = v_i x_i$ and $c'(t) = c_i'(t)x_i$. It follows that

$$\begin{aligned} \frac{D}{dt}(V) &= \frac{D}{dt}(v_i x_i) \\ &= \frac{dv_i}{dt}x_i + v_i\frac{D}{dt}x_i \\ &= \frac{dv_i}{dt}x_i + \sum_{i,j} v_i \cdot c_j' \nabla_{x_j} x_i \end{aligned}$$

Hence, we have

$$\frac{D}{dt}(V) = \sum_i \frac{dv_i}{dt} \cdot x_i + \sum_{i,j} v_i \cdot c_j' \nabla_{x_j} x_i.$$

which shows that the covariant derivative is uniquely defined by V and $c(t)$.

Existence: Define the covariant derivative $\frac{D}{dt}(V)$ as above. One should check conditions (i), (ii), (iii) are satisfied. □

3.2.2 Parallel vector fields

Definition 3.11. Let M be a differential manifold, ∇ be an affine connection, c be a smooth curve, and V be a vector field along the curve c. We say that V is **parallel** along c if $\frac{D}{dt}(V) = 0$ for all $t \in I$.

Remark. The above definition of parallel generalizes the notion of being parallel in the Euclidean setting.

Proposition 3.12. *Let M be a differential manifold, let ∇ be an affine connection, $c : I \to M$ be a smooth curve, and let $V_0 \in T_{c(0)}M$. Then there exists a unique parallel vector field V along c such that $V(t_0) = V_0$.*

Proof. In local coordinates, a vector field V is parallel if and only if

$$\sum_k \left(\frac{dv_k}{dt} + \sum_{i,j} v_i c_j' \cdot \Gamma_{ij}^k \right) x_k = 0$$

which yields a linear system

$$\frac{dv_i}{dt} = -\sum_{i,j} v_i \cdot c_j' \Gamma_{ij}^k$$

with the initial value

$$v_k(t_0') = v_k'.$$

Hence, our problem turns to a differential equation. From local existence and uniqueness, there exists $\epsilon > 0$ such that $V(t)$ is a parallel vector field on $(t_0' - \epsilon, t_0' + \epsilon)$ with initial value $V(t_0') = V_0' \in T_{c(t_0')}M$. Note that above is only local statement. For the global statement, consider the interval $[a, b] \subset I$, from local uniqueness it follows a parallel vector field $V(t)$ with $V(t_0) = V_0$ defined on $[a, b]$. As the interval $[a, b]$ is arbitrary, we can extend $V(t)$ to any value inside of the interval I. □

Definition 3.13. From the above proposition, we call $V(t)$ the **parallel transport** of V_0 along c.

Proposition 3.14. *The parallel transport defines an isomorphism from $T_{c(t_0)}M$ to $T_{c(T)}M$ for any $T \in I$.*

Proof. We sketch the proof. We define the function

$$\begin{aligned}\phi :& T_{c(t_0)}M \to T_{c(T)}M \\ & V_0 \mapsto \phi(V_0) = V(T)\end{aligned}$$

where $V(t)$ is a parallel vector field along the curve c such that $V(t_0) = V_0$. The linearity of ϕ follows directly from the linearity of $V(t)$. We need only to show that ϕ has an inverse. Consider the curve $\tilde{c} : I' \to M$ such that $\tilde{c}(t) = c(T - t)$. It follows that $\tilde{c}(0) = c(T)$. Set $\tilde{V}(t) = V(T - t)$. Clearly, $\tilde{V}$ also satisfies the above differential equation and $\tilde{V}(0) = V(T)$. Hence, this defines a map

$$\tilde{\phi} : T_{c(T)}M \to T_{c(t_0)}M$$

defined by parallel transport along $\tilde{c}$. It follows that $\tilde{\phi} = \phi^{-1}$. Therefore, ϕ is an isomorphism. □

Definition 3.15. Let (M, g) be a Riemannian metric and ∇ be an affine connection. We say that ∇ is **compatible** with the metric g if and only if for all smooth curves $c : I \to M$ and parallel vector fields V and W along c, we have $g(V, W) =$ constant.

Proposition 3.16. *Let (M, g) be a Riemannian manifold and ∇ be an affine connection. Then, ∇ is compatible with the g if and only if for all smooth curves $c : I \to M$ and vector fields V and W along c, we have*

$$\frac{d}{dt} g(V, W) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right) \tag{12}$$

Proof. $(\Rightarrow)$: Let $c : I \to M$ be a smooth curve and let $t_0 \in I$. Let $e_1, ..., e_n$ be a basis for $T_{c(t_0)}M$. Let $E_i(t)$ be the parallel vector field along c such that $E_i(t_0) = e_i$. Proposition **??** gives that $E_i(t)$ is a basis for $T_{c(T)}M$. Then, $V = V_i E_i$ and $W = W_i E_i$ for some $V_i, W_i : I \to \mathbb{R}$ that are C^∞. Hence, $g(V, W) = \sum_{i,j} V_i W_j g(E_i, E_j)$. How do we make this cleaner? Assume that the previous basis $\{e_i\}$ is also *orthonormal*. It follows that $g(E_i, E_j) = \delta_{ij}$ and $g(V, W) = \sum_i V_i W_i$. Hence,

$$\begin{aligned}\frac{d}{dt} g(V, W) &= \sum_i \frac{dV_i}{dt} W_i + \sum_i V_i \frac{dW_i}{dt} \\ &= g\left(\sum_i \frac{dV_i}{dt} E_i, \sum_j \frac{dW_j}{dt} E_j\right) + ... \\ &= g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right)\end{aligned}$$

□

Corollary 3.17. *Let (M, g) be a Riemannian manifold and let ∇ be an affine connection. Then, ∇ is compatible with g if and only if for any vector fields X, Y, Z on M, we have*

$$Xg(Y, Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z). \tag{13}$$

Proof. (⇒) Let X, Y, Z be vector fields on M and let $p \in M$. Consider the smooth curve $c : (-\epsilon, \epsilon) \to M$ such that $c(0) = p$ and $c'(0) = X(p)$. Hence,

$$\begin{aligned} Xg(Y,Z)(p) &= X_p(g(Y,Z)) \\ &= \frac{d}{dt}(g(Y,Z)|_{c(t)})_{t=0} \end{aligned}$$

□

3.2.3 Torsion

Definition 3.18. We define the **torsion** of an affine connection ∇ to be the map

$$T : \Gamma(M) \times \Gamma(M) \to \Gamma(M)$$

given by

$$T(X,Y) = \nabla_X Y - \nabla_X Y - [X,Y]. \tag{14}$$

Note that $T \in \Gamma(TM \otimes T^*M \otimes T^*M)$.

Remark. This means that T is a $(1,2)-$tensor and hence $(T(X,Y))_p$ depends *only* on the values of X and Y at the point p.

Definition 3.19. We say that an affine connection ∇ is **torsion-free** if and only if

$$\nabla_X Y - \nabla_Y X = [X,Y] \tag{15}$$

for all $X, Y \in \Gamma(M)$.

3.3 The Levi-Civita Connection

Theorem 3.20. ***(Levi-Civita)*** *Let (M,g) be a Riemannian manifold. Then, there exists a unique connection ∇ on M that satisfies*

1. *∇ is compatible with the Riemannian metric g*
2. *∇ is torsion-free.*

*We call this connection the **Levi-Civita connection** of the Riemannian manifold.*

Proof. We begin with the proof of uniqueness. If we suppose that such a connection ∇ exists, then for $X, Y, Z \in \Gamma(M)$ we have the following three equations which follow from compatibility with the metric:

$$\begin{aligned} Xg(Y,Z) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Yg(Z,X) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Zg(X,Y) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

Adding the first two equations and subtracting the third yields

$$\begin{aligned}Xg(Y,Z)+Yg(Z,X)-Zg(X,Y) &= g(\nabla_X Y+\nabla_Y X, Z)+g(\nabla_X Z-\nabla_Z X, Y)\\ &\quad + g(\nabla_Y Z-\nabla_Z Y, X)\end{aligned}$$

Using the second condition that ∇ is torsion free yields

$$2g(\nabla_X Y, Z) = Xg(Y,Z)+Yg(Z,X)-Zg(X,Y)-g([X,Z],Y)+g([X,Y],Z)-g([Y,Z],X)$$

which is true for all vector fields X, Y, Z. If we assume that g is non-generate, we have that $\nabla_X Y$ is uniquely defined by the metric g. This shows the uniqueness claim of the theorem. To show existence, we need only check that the equation derived above satisfies the conditions of being compatible with the metric and torsion free. □

3.3.1 The Christoffel symbols in local coordinates

In local coordinates, consider the chart (U,φ) with basis $x_i = D\varphi^{-1}\left(\frac{\partial}{\partial x_i}\right)$. We know that

$$\nabla_{x_i} y_i = \sum_k \Gamma_{ij}^k.$$

So,

$$\begin{aligned}g(\nabla_{x_i} y_i, x_k) &= \sum_\ell \Gamma_{ij}^\ell g_{\ell k}\\ &= \frac{1}{2}\left(x_i(g_{jk})+x_j(g_{ik})-x_k(g_{ij})\right).\end{aligned}$$

To rid ourselves of the metric on the right hand side of the first equality, we consider the *inverse metric* denoted by $\left(g^{km}\right)_{k,m}$ to be the inverse matrix of $(g_{ij})_{i,j}$. Multiplying by g^{km} on the right hand side yields

$$\sum_k\sum_\ell \Gamma_{ij}^\ell g_{\ell k} g^{km} = \sum_k \frac{1}{2} g^{km}\left(x_i(g_{jk})+x_j(g_{ik})-x_k(g_{ij})\right).$$

Switching the order of summation yields the following formula for the Christoffel symbols:

$$\Gamma_{ij}^\ell = \frac{1}{2}\sum_k g^{k\ell}\left(x_i(g_{jk})+x_j(g_{ik})-x_k(g_{ij})\right). \tag{16}$$

Example 3.21. We consider the Riemannian manifold $(\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. Then $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right\rangle = \delta_{ij}$. Hence, the Christoffel symbols in this setting are all zero. If we consider arbitrary vector fields X and Y, they have decomposition

$$X = a_i\frac{\partial}{\partial x_i} \qquad Y = b_j\frac{\partial}{\partial y_j}.$$

We calculate the Levi-Civita connection as

$$\nabla_X Y = \sum_{i,j} a_i \frac{\partial b_j}{\partial x_i} \frac{\partial}{\partial x_j}.$$

Finally, we turn to parallel vector fields in this setting. Let $c : I \to \mathbb{R}^n$ be a smooth curve and V be a parallel vector field along c. V has the decomposition $V = v_i \dfrac{\partial}{\partial x_i}$. It follows that

$$\begin{aligned} 0 = \frac{DV}{dt} &= \sum_i \left(\frac{dv_i}{dt} + \sum_k v_k c_j' \underbrace{\Gamma^i_{jk}}_{=0} \right) \frac{\partial}{\partial x_i} \\ &= \frac{dv_i}{dt} \end{aligned}$$

Hence, $v_i(t)$ is constant and $V(t) = V_0$.

4 Geodesics

We know how to define lines in $\mathbb{R}^n$. We now embark on how to generalize the notion of a line to a general Riemannian manifold. This is called a geodesic.

Definition 4.1. Let (M, g) be a Riemannian manifold. A smooth path $\gamma : I \to M$ is called a **geodesic** if and only if

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0 \tag{17}$$

for every $t \in I$.

Remark.

1. We have the calculation from compatibility of the metric g:

$$\frac{d}{dt} g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = g\left(\frac{D}{dt}\left(\frac{d\gamma}{dt}\right), \frac{d\gamma}{dt}\right) + g\left(\frac{d\gamma}{dt}, \frac{D}{dt}\left(\frac{d\gamma}{dt}\right)\right) = 0.$$

   Hence, if γ is a geodesic, the quantity $|\gamma'(t)|$ is constant.

2. We call $\gamma(I)$ the geodesic with an abuse of notation.

3. If $|\gamma'(t)| = 1$, we say that γ is parametrized with respect to arc length: $s(t) = \displaystyle\int_{t_0}^{t} |\gamma'(s)| ds$

4.1 The geodesic equation

We investigate equation (17) in local coordinates. Let (U, φ) be a local chart and $p \in U$. We define $\tilde{\gamma}$ as the map from I to $\varphi(U)$ and we have $\tilde{\gamma}(t) = (\gamma_1(t), ..., \gamma_n(t))$. It follows that

$\frac{d\tilde{\gamma}}{dt} = \gamma_i'(t)\frac{\partial}{\partial x_i}$. Substituting into equation (17), we have

$$\begin{aligned}\frac{D}{dt}\left(\gamma_i'(t)\frac{\partial}{\partial x_i}\right) &= \frac{d^2\gamma_i}{dt^2}\cdot\frac{\partial}{\partial x_i} + \frac{d\gamma_i}{dt}\cdot\frac{D}{dt}\left(\frac{\partial}{\partial x_i}\right)\\ &= \frac{d^2\gamma_i}{dt^2}\cdot\frac{\partial}{\partial x_i} + \frac{d\gamma_i}{dt}\cdot\nabla_{\frac{d\tilde{\gamma}}{dt}}\left(\frac{\partial}{\partial x_i}\right)\\ &= \frac{d^2\gamma_i}{dt^2}\cdot\frac{\partial}{\partial x_i} + \frac{d\gamma_i}{dt}\cdot\frac{d\gamma_j}{dt}\nabla_{\frac{\partial}{\partial x_j}}\left(\frac{\partial}{\partial x_i}\right)\end{aligned}$$

and hence

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \sum_k\left(\frac{d^2\gamma_k}{dt^2} + \sum_{i,j}\frac{d\gamma_i}{dt}\cdot\frac{d\gamma_j}{dt}\Gamma_{ij}^k\right)\frac{\partial}{\partial x_k} \tag{18}$$

We therefore have the following proposition.

Proposition 4.2. *Let (M, g) be a Riemannian manifold and $\gamma : I \to M$ be a smooth curve. Then, γ is a geodesic if and only if the following equation holds*

$$\frac{d^2\gamma_k}{dt^2} = -\sum_{i,j}\frac{d\gamma_i}{dt}\frac{d\gamma_j}{dt}\Gamma_{ij}^k. \tag{19}$$

Equation (19) yields that being a geodesic equates to a second order ODE in local coordinates. We would like to see when solutions to equation (19) exist. First, since the geodesic equation is second order, we will need two initial conditions to say something about existence. We have the following two initial conditions where t_0 is some initial time:

$$\begin{aligned}\gamma_i(t_0) &= p_0\\ \frac{d\gamma}{dt}(t_0) &= v_0,\end{aligned}$$

where p_0 is the initial position and v_0 is the initial velocity. We have that $(\tilde{\gamma}(t_0), \tilde{\gamma}(t_0)) \in T(\varphi(U)) \cong \varphi(U) \times \mathbb{R}^n$. We are interested in the existence and uniqueness in a domain $(t_0 - \epsilon, t_0 + \epsilon)$. We have the following theorem.

Theorem 4.3. *Let (M, g) be a Riemannian metric. For any $p_0 \in M$ and $v_0 \in T_{p_0}M$, there exists an open neighborhood $U' \subset M$ that contains p_0 and $U' \times V'$ which is a neighborhood of $(p_0, v_0) \in TM$ and $\epsilon > 0$ such that there exists an unique solution to equation (19) $\gamma_{p,v} : (-\epsilon, \epsilon) \to M$ satisfying the initial conditions $\gamma_{p_0,v_0}(0) = p_0$ and $\gamma_{p_0,v_0}'(0) = v_0$ for any $(p, v) \in U' \times V'$. Moreover,*

Lemma 4.4. *(Homogeneity of geodesics). If the geodesic γ_{p_0,v_0} is defined on the interval $(-\epsilon, \epsilon)$, then the smooth curve γ_{p,av_0} defined by*

$$\gamma_{p_0,av_0} = \gamma_{p_0,v_0}(at)$$

defined on $(-\frac{\epsilon}{a}, \frac{\epsilon}{a})$ is also a geodesic.

Proof. □

We now present the geometric perspective of Theorem 4.3.

Theorem 4.5. *(Local Existence and Uniqueness). Let (M, g) be a Riemannian manifold and $p_0 \in M$. Then there exists an open set $U \subset M$, $\epsilon > 0$, and $\delta > 0$ such that for all $p \in U$, $v \in T_pM$, and $|v| < \delta$, there exists a unique solution to equation (19)*

$$\gamma_{p,v} : (-\epsilon, \epsilon) \to M$$

such that $\gamma_{p,v}(0) = p$ and $\gamma'_{p,v}(0) = v$. Moreover, if we let $\mathcal{U} = \{(p, v) \mid p \in U, v \in T_pM, |v| < \delta\}$, then $\gamma : \mathcal{U} \times (-\epsilon, \epsilon) \to M$ where $\gamma(p, v, t) = \gamma_{p,v}(t)$ is smooth.

In practice, how do we solve second-order differential equations? We usually write them as a state space system. Ergo, let $X_i = \gamma_i$ and $Y_i = \dfrac{dX_i}{dt}$. Then, equation (19) becomes the following system of first-order differential equations:

$$\frac{dX_k}{dt} = Y_k$$
$$\frac{dY_k}{dt} = -\sum_{i,j} \Gamma^k_{ij} Y_i Y_j$$

The solution is given by

$$\alpha : I \to TM$$
$$t \mapsto \alpha(t) = \left(\gamma(t), \frac{d\gamma}{dt}\right)$$

where α is an integral curve for the vector field G on the tangent bundle TM. In local coordinates,

$$G(x, y) = \left(xy, y_i - \sum_{ij} \Gamma^k_{ij} y_i y_j\right).$$

Proposition 4.6. *The exists a unique vector field G on the tangent bundle TM such that the integral curves after projection on M are geodesics on M.*

Definition 4.7. This vector field G is called the **geodesic vector field** on TM and its flow is called the **geodesic flow** on TM.

4.2 The exponential map

Let $p \in M$ and $\mathcal{U}$ be the same as in Theorem **??**. Then, we define the **exponential map**

$$\exp : \mathcal{U} \to M$$

be given by

$$\exp_q(v) = \gamma(1, q, v) = \gamma\left(|v|, q, \frac{v}{|v|}\right) \tag{20}$$

for $(q, v) \in \mathcal{U}$.

Geometrically, $\exp_q(v)$ is a point in M obtained by going out the length equal to $|v|$ starting from q, along the geodesic which passes through q whose velocity is equal to $\dfrac{v}{|v|}$.

Proposition 4.8. *Let M be a differentiable manifold. Given $p \in M$, there exists $\delta > 0$ such that*

$$\exp_p : B_\delta(0) \subset T_pM \to M$$

is a local diffeomorphism.

Proof. We calculate $D(\exp_p)_0(v)$:

$$\begin{aligned} D(\exp_p)_0(v) &= \frac{d}{dt}(\exp_p(tv))\Big|_{t=0} \\ &= \frac{d}{dt}\gamma_{p,tv}(1)\Big|_{t=0} \\ &= \frac{d}{dt}\gamma_{p,v}(t)\Big|_{t=0} \\ &= v \end{aligned}$$

which yields that $D(\exp_p)_0 = \text{id}$. It follows from the inverse function theorem that $\exp_q$ is a local diffeomorphism on a neighborhood of 0. □

Definition 4.9. If $V \subset T_pM$ is a neighborhood of the vector 0 such that the exponential map $\exp_p : V \to U \subset M$ is a diffeomorphism, then U is called a **normal neighborhood** of p. Moreover, $U = \exp_p(B_\delta(0))$ is called the **geodesic normal ball of radius** δ.

Proposition 4.10. *Let (M, g) be a Riemannian manifold. For all $p \in M$, there exists $W \subset M$ which is a neighborhood of p and $\delta > 0$ such that for all $q \in W$, the exponential map $\exp_q$ is a diffeomorphism of $B_\delta(0) \subset T_qM$ onto $\exp_q(B_\delta(0))$ which contains W.*

Corollary 4.11. *For all $p \in M$, there exists a neighborhood W of p and $\delta > 0$ such that for all $x, y \in W$, thee exists a unique vector $v \in T_xM$ with $|v| < \delta$ such that $\exp_x v = y$.*

4.3 Minimizing properties of geodesics

Definition 4.12. A **piecewise differentiable curve** is a continuous map $c : [a, b] \to M$ of a closed interval $[a, b] \subset \mathbb{R}$ into M satisfying the following condition: there exists a partition $a = t_0 < t_1 < \cdots < t_{k-1} < t_k = b$ of $[a, b]$ such that

Definition 4.13. Let A be a connected subset of $\mathbb{R}^2$ such that $\partial A = \overline{A} \backslash A^\circ$ is a piecewise differential curve with vertex angle different than π. A **parametrixed surface** in M is a differentiable map from $S : A \to M$. Note that S admits a smooth extension in an open neighborhood of A.

Definition 4.14. A vector field V along S is a smooth map which associates to each point $x = x(u, v) \in A$ a vector $V(x) \in T_{S(x)}M$.

We use the following notation for derivatives

$$Ds_x\left(\frac{\partial}{\partial u}\right) = \frac{\partial s}{\partial u}$$
$$Ds_x\left(\frac{\partial}{\partial v}\right) = \frac{\partial s}{\partial v}$$

Definition 4.15. If V is a vector field along $S : A \to M$, then we denote $\dfrac{DV}{\partial u}$ as the covariant derivative of the vector field V along the curve

$$S(\cdot, v_0) : (u_0 - \epsilon, u_0 + \epsilon) \to M.$$

Similarly, we denote $\dfrac{DV}{\partial v}$ as the covariant derivative of the vector field V along the curve

$$S(\cdot, u_0) : (u_0 - \epsilon, u_0 + \epsilon) \to M.$$

Note that these are both vector fields along S.

Lemma 4.16. *Let M be a differentiable manifold, ∇ be a symmetric connection, and $S : A \to M$ be a parametrized surface. Then*

$$\frac{D}{\partial v}\left(\frac{\partial s}{\partial u}\right) = \frac{D}{\partial u}\left(\frac{\partial s}{\partial v}\right) \tag{21}$$

We now discuss what it means to minimize a geodesic. First, we have to understand how to calculate the length of curves in a Riemannian manifold. We have the following definitions:

Definition 4.17. Let (M, g) be a Riemannian manifold and c be a piecewise smooth curve. Define the **length** of c as

$$\ell(c) := \int_a^b |c'(t)| \; dt \tag{22}$$

where $|c'(t)| = \sqrt{g_{c'(t)}\left(\dfrac{dx}{dt}, \dfrac{dc}{dt}\right)}$.

Definition 4.18. We say that $\gamma : [a, b] \to M$ is length minimizing if, for any piecewise smooth curve c such that $c(a) = \gamma(a)$ and $c(b) = \gamma(b)$, $\ell(c) \geq \ell(\gamma)$.

We now present the following theorem on local length minimizing.

Theorem 4.19 (Local length minimizing)**.** *Let (M, g) be a Riemannian manifold $p \in M$. Let $B = B_r(p) \subset M$ be a geodesic normal ball and let $\gamma : [0, 1] \to B$ be a geodesic such that $\gamma(0) = p$. If $c : [0, 1] \to M$ is any piecewise differentiable curve such that $c(0) = p$ and $c(1) = \gamma(1)$, then $\ell(c) \geq \ell(\gamma)$. Moreover, if equality holds, then $\gamma([0, 1]) = c([0, 1])$.*

Proof. Assume that $c([0,1]) \subset B$ and $p \in c([0,1])$ (otherwise, consider a smaller piece). Let $v : (0,1] \to TM$ be defined such that $|v(t)| = 1$ and let $r : [0,1] \to \mathbb{R}_{\geq 0}$ be defined such that $c(t) = \exp_p(r(t) \cdot v(t))$. We have that both v and r are piecewise differentiable function form some partition; call this by $0 = t_0 < \cdots t_k = 1$. Now, consider the function $f : [0,R] \times [0,1] \to M$ defined by $f(r,t) = \exp_p(r \cdot v(t))$, which is a parametrized surface. Then, $c(t) = f(r(t),t)$. If $t \neq t_i$, then

$$\begin{aligned}\frac{dc}{dt} &= \frac{df}{dr}\frac{dr}{dt} + \frac{df}{dt} \\ &= (D\exp_p)_{r,v(t)} v(t) \cdot \frac{dr}{dt} + (D\exp_p)_{r,v(t)}(r \cdot v'(t))\end{aligned}$$

and hence

$$\begin{aligned}\left|\frac{dc}{dt}\right|^2_{c(t)} &= g_{c(t)}\left(\frac{dc}{dt}, \frac{dc}{dt}\right) \\ &= g_{c(t)}\big((D\exp_p)_{r,v(t)} v(t), (D\exp_p)_{r,v(t)} v'(t)\big) \cdot \left(\frac{dr}{dt}\right)^2 \\ &\quad + 2g_{c(t)}\big((D\exp_p)_{r,v(t)} v(t), (D\exp_p)_{r,v(t)} v'(t)\big) + \left|\frac{df}{dt}\right|^2_{c(t)} \\ &= |v(t)|^2_{g_p} \cdot \left(\frac{dr}{dt}\right)^2 + 0 + \left|\frac{df}{dt}\right|^2_{c(t)}\end{aligned}$$

Where the last equality follows from Gauss' lemma and the zero term comes from

$$g(v(t), v'(t)) = \frac{1}{2}\frac{d}{dt} g(v(t), v(t)) = 0.$$

It follows that

$$\begin{aligned}\ell(c|_{[t_i,t_{i+1}]}) &= \int_{t_i}^{t_{i+1}} |c'(t)|_{c(t)}\, dt \\ &= \int_{t_i}^{t_{i+1}} \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(\frac{df}{dt}\right)^2} dt \\ &\geq \int_{t_i}^{t_{i+1}} \sqrt{\left(\frac{dr}{dt}\right)^2} dt \\ &\geq \int_{t_i}^{t_{i+1}} \frac{dr}{dt} dt \\ &= r(t_{i+1}) - r(t_i).\end{aligned}$$

Hence,

$$\begin{aligned}\ell(c) &\geq \sum_{i=1}^{k-1} r(t_{i+1}) - r(t_i) \\ &= r(1) - r(0) \\ &= |\gamma'(t)| \\ &= \ell(\gamma).\end{aligned}$$

Finally, if equality holds, using the inequalities above, we have that $\left|\frac{dt}{dt}\right| = 0$. It follows that $v(t)$ is constant that is equal to $v(1)$. Therefore, $c(t)$ parametrized the geodesic $\gamma([0,1])$. □

Corollary 4.20. *If a piecewise differentiable curve $c : [a,b] \to M$ is parametrized with respect to arc length is minimizing the length of all piecewise differentiable curves from $c(a)$ to $c(b)$, then c is a geodesic and smooth for all $t \in [a,b]$.*

4.3.1 Riemannian manifolds as metric spaces

Definition 4.21. Let (M,g) be a connected Riemannian manifold. For any $p,q \in M$, we define

$$d(p,q) \coloneqq \inf_c \{\ell(c) \mid c : [a,b] \to M \text{ such that } c \in C^\infty([a,b]),\ c(a) = p,\ c(b) = q\}$$

is called the **distance from** p **to** q.

Proposition 4.22. *Let (M,g) be a Riemannian manifold and d be defined as above. Then,*

1. *(M,d) is a metric space, and*
2. *the metric topology on M coincides with the underlying top of the smooth manifold $<$.*

Proof. The proof follows from checking the three properties of a metric. □

4.3.2 Geodesic completeness

Definition 4.23. A Riemannian manifold (M,g) is called **geodesically complete** if, for any $p \in M$, the geodesic $\gamma(t)$ starting at p is defined for all values of the parameter $t \in \mathbb{R}$.

Remark.
1. If M is geodesically complete, then the exponential map is well defined.
2. The space $(\mathbb{R}^2 - \{0\}, \langle\cdot,\cdot\rangle_{Euc})$ is *not* geodesically complete.

Theorem 4.24 (Hopf-Rimow)**.** *Let (M,g) be a connected Riemannian manifold and $p \in M$. Then the following two statements are equvalent:*

1. *If the exponential map at p is defined on the entire tangent space T_pM, then any point $q \in M$ can be joint to p by a length minimizing geodesic.*

2. *If the manifold is geodesically complete, then any two points can be joint by a minimal geodesic.*

Remark. Note that minimizing geodesics are *not* unique. For an example, consider the $n-$sphere.

Proposition 4.25. *Let (M, g) be a Riemannian manifold. Then the following are equivalent:*

1. *(M, g) is geodesically complete*
2. *For any $p \in M$, the exponential map is defined on the entire tangent space T_pM*
3. *There exists $p \in M$ such that the exponential map is defined on the tangent space T_pM*
4. *Any closed and bounded subset of the manifold is compact.*
5. *The metric on the manifold is complete.*

Corollary 4.26. *If the manifold M is compact, then M is complete in the geodesic sense and as a metric.*

5 Curvature

5.1 The covariant derivative of a tensor

Recall that a tensor of type (p, q) is a section of the form

$$T^{p,q}M = \underbrace{TM \otimes \cdots \otimes TM}_{p} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{q}.$$

We use the notation $T \in \Gamma(T^{p,q}M)$ to signal that T is a (p, q) tensor. Here are some common examples of tensors:

Example 5.1.

Example 5.2. For a Riemannian manifold (M, g), the associated Riemannian metric g is also a tensor. More specifically, g is a $(0, 2)$ tensor and hence $g \in \Gamma(T^{0,2}M)$.

Definition 5.3. Given a Riemannian manifold (M, g), a **tensor contraction** is a map

$$c : \Gamma(TM \otimes T^*M) \to C^\infty(M)$$

defined by

$$(X, \alpha)_p \mapsto \alpha_p(X_p). \tag{23}$$

In general, a tensor contraction is a map

$$c_{ij} : \Gamma(T^{p,q}M) \to \Gamma(T^{p-1,q-1}M)$$

defined by

$$(X_1 \otimes \cdots \otimes X_p \otimes \alpha_1 \otimes \cdots \otimes \alpha_q) \mapsto X_1 \otimes \cdots \otimes \hat{X}_i \otimes \cdots \otimes X_p \otimes \alpha_1 \otimes \cdots \otimes \hat{\alpha_j} \otimes \cdots \otimes \alpha_q. \tag{24}$$

Let E be a finite dimensional vector space, then there is an isomorphism

$$E^* \otimes E \xrightarrow{\sim} \mathrm{End}(E) = \{L : E \to E \mid L \text{ linear}\}$$

given by

$$\alpha \otimes v \mapsto L_{\alpha,v}(u) = \alpha(u) \cdot v$$

and the contraction c corresponds to the trace in the following diagram

$$\begin{array}{ccc} E^* \otimes E & \longrightarrow & \mathrm{End}(E) \\ & \underset{c}{\searrow} & \downarrow \mathrm{tr} \\ & & \mathbb{R} \end{array}$$

Proposition 5.4. *Let (M, g) be a Riemannian manifold and $X \in \Gamma(TM)$. Then, the covariant derivative has a unique extension $(\nabla_X : \Gamma(TM) \to \Gamma(TM))$ to endomorphisms of $\Gamma(T^{p,q}M)$ defined by*

$$\nabla_X : \Gamma(T^{p,q}M) \to \Gamma(T^{p,q}M)$$

which satisfies the following conditions:

1. *for all $s \in \Gamma(T^{p,q}M)$ with $p, q > 0$ and any contraction c_{ij} then $\nabla_X(c_{ij}(s)) = c_{ij}(\nabla_X s)$*
2. *for any S, T tensors, $\nabla_X(S \otimes T) + \nabla_X S \otimes T = S \otimes \nabla_X T$*

Proof. Let $\alpha \in \Gamma(T^{0,1}M)$ and $Y \in \Gamma(T^{1,0}M)$, then $\nabla_X(Y \otimes \alpha) = \nabla_X Y \otimes \alpha + Y \otimes \nabla_X \alpha$. Applying the contraction c to both sides, we have

$$c(\nabla_X(Y \otimes \alpha)) = c(\nabla_X Y) + c(Y \otimes \nabla_X \alpha).$$

Expanding the left hand side,

$$\begin{aligned} c(\nabla_X(Y \otimes \alpha)) &= \nabla_X(c(Y \otimes \alpha)) \\ &= \nabla_X(\alpha(Y)) \\ &= X(\alpha(Y)). \end{aligned}$$

Similarly, for the right hand side we have

$$c(\nabla_X Y \otimes \alpha) = \alpha(\nabla_X Y)$$

and

$$c(Y \otimes \nabla_X \alpha) = \nabla_X \alpha(Y).$$

It follows that

$$X(\alpha(Y)) = \alpha(\nabla_X Y) + (\nabla_X \alpha)(Y).$$

Then, the covariant derivative $\nabla_X : \Gamma(T^{0,1}M) \to \Gamma(T^{1,0}M)$ is uniquely defined buy

$$(\nabla_X \alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y)$$

for any $Y \in \Gamma(TM)$. In general, $\omega \in \Gamma(T^{0,p}M)$. Then, for $(X_1, ..., X_p) \in \Gamma(TM)$

$$(\nabla_X \omega)(X_1, ..., X_p) = X \cdot (\omega(X_1, ..., X_p)) - \sum_{i=1}^{p} \omega(X_1, ..., \nabla_X X_i, ..., X_p)$$

□

Let g be a Riemannian metric. As we discussed above, g is a $(0, 2)$ tensor. We take the covariant derivative of g as follows

$$(\nabla_X g)(X, Y) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z).$$

Note that the affine connection ∇ is compatible with the metric g which is equivalent to $(\nabla_X g) = 0$ for all $X \in \Gamma(TM)$.

5.2 Properties of curvature

From the previous section, we define the curvature tensor R in the following way.

Definition 5.5. Let (M, g) be a Riemannian manifold and consider vector fields $X, Y \in \Gamma(M)$. Let

$$R(X, Y) : \Gamma(TM) \to \Gamma(TM)$$

be defined by

$$R(X, Y)(Z) \coloneqq \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z \tag{25}$$

for all $Z \in \Gamma(TM)$. R is called the **curvature tensor** on (M, g). It is a $(1, 3)$ tensor.

5.2.1 Local coordinates

We consider the above definition of the curvature tensor in local coordinates. Consider the local chart (U, φ) on a Riemannian manifold (M, g). Let $x_i = \varphi^* \left(\frac{\partial}{\partial x_i} \right)$ be a basis for the vector field. Consider the vector fields $X = a_i X_i$, $Y = b_j Y_j$, and $Z = c_k Z_k$. It follows that

$$R(X, Y)Z = \sum_{i,j,k} a_i b_j c_k \underbrace{R(X_i, Y_j)Z_k}_{= \sum_m R^m_{ijk} X_m}.$$

We call R^m_{ijk} as the **coefficient function of the curvature with respect to the local chart**. We express the coefficient function R in the following manner:

$$\begin{aligned} R(X_i, X_j)X_k &= \nabla_{X_j} \left(\sum_\ell \Gamma^\ell_{ij} X_\ell \right) - \nabla_{X_i} \left(\sum_\ell \Gamma^\ell_{jk} X_\ell \right) - \underbrace{\nabla_{[X_i, X_j]} X_k}_{= 0} \\ &= \sum_\ell \left(X_j \left(\Gamma^\ell_{ik} \right) - X_i \left(\Gamma^\ell_{jk} \right) \right) \cdot X_\ell + \sum_\ell \sum_m \Gamma^\ell_{ik} \Gamma^m_{j\ell} \cdot X_m \end{aligned}$$

5.2.2 The curvature operator

Definition 5.6. Let (M,g) be a Riemannian manifold and consider the following $(0,4)$ tensor. Define

$$\mathcal{R} : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

given by

$$\mathcal{R}(X,Y,Z,T) = g\left(R(X,Y)Z,T\right) \tag{26}$$

as the equivalent $(0,4)$**curvature tensor**.

This definition leads to the following properties of the new curvature tensor.

Proposition 5.7. *Given vector fields $X,Y,Z,T \in \Gamma(TM)$, the curvature tensor $\mathcal{R}$ satisfies the following properties:*

1. *$\mathcal{R}$ is a $(0,4)$ tensor*
2. *$\mathcal{R}(X,Y,Z,T) + \mathcal{R}(Y,Z,X,T) + \mathcal{R}(Z,X,Y,T) = 0$ (called Bianchi's first identity)*
3. *$\mathcal{R}(X,Y,Z,T) = -\mathcal{R}(Y,X,Z,T)$*
4. *$\mathcal{R}(X,Y,Z,T) = -\mathcal{R}(X,Y,T,Z)$*
5. *$\mathcal{R}(X,Y,Z,T) = \mathcal{R}(Z,T,X,Y)$*

Proof. □

So far, we have considered two different forms of curvature: the first is a $(1,3)$ tensor and the second is a $(0,4)$ tensor. Using the latter type of curvature and the properties in Proposition 5.7, we have $\mathcal{R}$ as the following map

$$\mathcal{R} : \Lambda^2(TM) \times \Lambda^2(TM) \to C^\infty(M)$$

which yields that we can write $\mathcal{R}$ as

$$\mathcal{R} : \mathrm{Sym}^2(\Lambda^2(TM)) \to C^\infty(M).$$

These representations of $\mathcal{R}$ give way to the following commutative diagram:

$$\begin{array}{ccc} \mathcal{R} : \Lambda^2(TM) & \longrightarrow & (\Lambda^2(TM))^* \\ & \searrow^{\rho} & \big\downarrow {\scriptstyle \text{isomorphism}} \\ & & \Lambda^2(TM) \end{array}$$

We call ρ the **curvature operator**.

Proposition 5.8. *The curvature operator ρ is self adjoint with respect to the induced metric on $\Lambda^2(TM)$.*

Remark. The above proposition is equivalent to property 5 in Proposition 5.7.

5.3 Sectional curvature

Definition 5.9. Let (M, g) be Riemannian manifold. For any point $p \in M$, let $\alpha \subset T_pM$ be a two dimensional subspace with $\{u, v\}$ as a basis for α.

Definition 5.10. For a Riemannian manifold (M, g) and two dimensional subspace $\alpha \subset T_pM$, we define the **sectional curvature** $k(\alpha)$ as

$$k(\alpha) = k(u, v) = \frac{R(u, v, u, v)}{|u \wedge v|^2} \tag{27}$$

where $|u \wedge v|^2 = |u|^2|v|^2 - (g_p(u, v))^2$

Proposition 5.11. *The sectional curvature does not depend on the choice of the basis for the two dimensional subspace, α.*

Proof. To avoid calculating, we observe that we can pass from the basis $\{u, v\}$ of the two dimensional subspace α to any other basis $\{u', v'\}$ by iterating the following elementary transformations:

1. $\{u, v\} \to \{v, u\}$
2. $\{u, v\} \to \{\lambda u, v\}$
3. $\{u, v\} \to \{u + \lambda v, v\}$.

It is easy to see that the sectional curvature $k(u, v)$ is invariant by such transformations. □

Proposition 5.12. *The sectional curvature determines the curvature tensor. That is,*

$$k(\alpha) \text{ for any } \alpha \subset T_pM \iff R(X, Y, Z, T)_p \text{ for all } X, Y, Z, T \in \Gamma(M).$$

Proof. □

Example 5.13. We want to find manifolds whose sectional curvature is constant. We consider two examples.

1. The first example is the Euclidean space with the Euclidean metric. We have shown that in $(\mathbb{R}^n, \langle\cdot,\cdot\rangle_{\text{Euc}})$, the Christoffel symbols $\Gamma^k_{ij} = 0$. It follows that $R^i_{jkk} = 0$ and hence the sectional curvature is zero.
2. (S^n, g_1) has sectional curvature that is a constant.

Proposition 5.14. *Let (M, g) be a Riemannian manifold. Then the following are equivalent:*

1. $k(\alpha) = k_0$ *for all* $p \in M$ *and all two-dimensional subspaces* $\alpha \subset T_pM$
2. $g(R(u, v)w, t) = k_0(g(u, v)g(v, t) - g(v, w)g(u, t))$
3.

Corollary 5.15. *Let (M, g) be a Riemannian metric, $p \in M$, $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM, and let $R_{ijk\ell} = g(R(e_i, e_j)e_k, e_\ell)$. Then there is constant sectional curvature if and only if $R_{ijk\ell} = k_0 (\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk})$.*

Proposition 5.16 (The Second Bianchi Identity)**.** *Let (M, g) be a Riemannian manifold $X, Y, Z, T, W \in \Gamma(TM)$ and R be a $(0, 4)$ curvature tensor. Then*

$$(\nabla_X R)(Y, Z, T, W) + (\nabla_Y R)(Z, X, T, W) + (\nabla_Z R)(X, Y, T, W) = 0 \tag{28}$$

where

$$\begin{aligned}(\nabla_X R)(Y, Z, T, W) = X(R(Y, Z, T, W)) - R(\nabla_X Y, Z, T, W) - R(Y, \nabla_X Z, T, W)\\ - R(Y, Z, \nabla_X T, W) - R(Y, Z, T, \nabla_X W)\end{aligned}$$

5.3.1 The geodesic frame

Proposition 5.17. *Let (M, g) be a Riemannian manifold and $p \in M$. Then there exists a neighborhood U of p and $E_1, ..., E_n \in \Gamma(U)$ such that $\{E_1(q), ..., E_n(q)\}$ is an orthonormal basis of the tangent space T_qM for any $q \in U$ and $\nabla_{E_i} E_j(p) = 0$.*

Proof. Let $U = B_\epsilon(p) \subset M$ for $\epsilon > 0$ small enough. Consider an orthonormal basis of the the tangent space T_pM $\{e_1, ..., e_n\}$. Define $E_i(q)$ from the parallel transport of e_i along the geodesic $\gamma(t) = \exp_p(tv)$ where $\exp_p(v) = q$. $\square$

Lemma 5.18 (Schur's Lemma)**.** *Let (M, g) be a Riemannian manifold whose dimension is greater than or equal to 3. Suppose that M is isotropic. That is, $k_p(\alpha) = f(p) =$ constant for some $f \in C^\infty(M)$. Then, M has constant sectional curvature.*

Example 5.19. Consider the Riemannian manifold (S_n^r, g_r) where $S_n^r = S_r(0)$ and g_r is the induced submanifold metric. Define the function

$$f : S_1 \to S_r$$

given by

$$f(x) = rx.$$

Consider the function defined by $h = (f^* g_r)$ where

$$\begin{aligned} h(u, v) &= g_r(df(u), df(v)) \\ &= g_r(ru, rv) \\ &= r^2 \underbrace{g_r(u, v)}_{\text{Euclidean metric}} \\ &= r^2 \langle u, v \rangle_r \\ &= r^2 g_1(u, v). \end{aligned}$$

Hence, $h = r^2 g_1$ which yields that the curvatures with respect to h and g_1 are equal. In local coordinates, the Christoffel symbols are equal. We can then relate the sectional curvatures between the two metrics in the following way:

$$\begin{aligned} k^h(\alpha) &= \frac{R^h(u,v,u,v)}{|u \wedge v|_h^2} \\ &= \frac{R^{g_1}(u,v,u,v)}{r^2 |u \wedge v|_{g_1}^2} \\ &= \frac{1}{r^2} k^{g_1}(\alpha) \end{aligned}$$

Example 5.20. We consider a second example: the curvature of a product on a Riemannian manifold. If we have two Riemannian manifolds: (M, g) and (N, h), we define the **product Riemannian manifold** as $(M \times N, g \times h)$. For $(p, q) \in M \times N$, we have that

$$T_{(p,q)} M \times N \cong T_p M \oplus T_q M$$

where the isomorphism is given by

$$T_{(p,q)} M \times N \ni u \to (d\pi_1(u), d\pi_2(u)).$$

If ∇^g and ∇^h are the Levi-Civita connections on M and N, respectively, and ∇ is the Levi-Civita connection on the product manifold $M \times N$, then $\nabla_X Y = \nabla^g_{X_1} Y_1 + \nabla^h_{X_2} Y_2$ where $X_i = (D\pi_i)X$ and $Y_i = (D\pi_i)Y$. As a consequence,

$$R(U, V, W, T) = R^g(U_1, V_1, W_1, T_1) + R^h(U_2, V_2, W_2, T_2)$$

for $U, V, W, T \in T_{(p,q)} M \times N$.

Remark. Note that $(S^2 \times S^2, g_1 \times g_2)$ does **not** have constant sectional curvature.

Conjecture 5.21 (The Hopf Conjecture)**.** *The product manifold $M = S^2 \times S^2$ does not admit a metric such that the sectional curvature is positive for every $p \in M$ and every two-dimensional subspace of $T_p M$.*

5.4 Ricci curvature and scalar curvature

Consider a $(1, 3)$ curvature tensor on a Riemannian manifold defined by

$$R : \Gamma(TM) \times \Gamma(TM) \times \Gamma(TM) \to \Gamma(TM)$$

given by

$$(X, Y, Z) \mapsto R(X, Y)Z.$$

Now, fix two of the vector fields $X, Z \in \Gamma(TM)$ and let the vector field Y be variable. That is,

$$R(X, \cdot)Z : \Gamma(TM) \to \Gamma(TM).$$

If we take the trace of the above map, we arrive at the **Ricci curvature**.

Definition 5.22. Let (M, g) be a Riemannian manifold and $X, Y \in \Gamma(TM)$. The **Ricci curvature** is defined as

$$Ric(X, Y) := \operatorname{tr}\left(R(X, \cdot)Z\right). \tag{29}$$

For $p \in M$, let $\{e_1, ..., e_n\}$ be an orthonormal basis of T_pM. Then we have the Ricci curvature of $X, Y \in \Gamma(TM)$ as

$$Ric(X, Y)_p = \sum_i g(R(X_p, e_i)Y_i, e_i)$$

Remark. Note that the above definition does not depend on the choice of the orthonormal basis.

We have that the Ricci curvature is a $(0, 2)$ tensor.

Lemma 5.23. *For vector fields $X, Y \in \Gamma(TM)$, we have*

$$Ric(X, Y) = Ric(Y, X).$$

It is the case that

$$Ric : \Gamma(TM) \times \Gamma(TM) \to C^\infty(M)$$

and at a point $p \in M$,

$$Ric_p : T_pM \times T_pM \to \mathbb{R}$$

It follows that $Ric_p \in T_p^*M \otimes T_p^*M$.

Definition 5.24. Let $S \in C^\infty(M)$. We define the **Ricci scalar curvature** at the point $p \in M$ as the sum of the Ricci curvature over an orthonormal basis of the tangent space at p. That is,

$$S(p) := \sum_i Ric(e_i, e_i) \tag{30}$$

where $\{e_i\}$ is an orthonormal basis for T_pM.

Proposition 5.25. *If dim(M)=3, then the Ricci tensor determines the full curvature tensor.*

Proof. Let $p \in M$ and $\{e_1, e_2, e_3\}$ be an orthonormal basis for the tangent space T_pM. Then, we calculate

$$Ric(e_1, e_2) = \sum_{i=1}^{3} g(R(e_1, e_i)e_2, e_i).$$

If $i = 1$ or $i = 2$, we have $R(e_1, e_1) = 0$ and $R(e_2, e_2) = 0$. It follows that only $i = 3$ determines the Ricci curvature. Hence,

$$Ric(e_1, e_2) = g(R(e_1, e_3)e_2, e_3).$$

We have that

$$R_{i,j,k,\ell}(p) = g_p(R(e_i, ej)e_k, e_\ell).$$

First, if $i = j$ or $k = \ell$, the curvature is zero. But, we have that $\{i, j, k, \ell\} \subset \{1, 2, 3\}$. If the cardinality of $\{i, j, k, \ell\} = 3$, then we have that

□

Definition 5.26. If the Ricci curvature is constant; that is,

$$Ric(X, Y) = \alpha g(X, Y)$$

for some $\alpha \in \mathbb{R}$, we call g the **Einstein metric**.

Remark. Note that there are topological and smooth obstructions to the existence of Einstein metrics. The existence is known for metrics in the Kahler-Einstein case. Here, M admits a complex structure and the metric has a local potential.

Proposition 5.27. *Let (M, g) be a Riemannian manifold such that parallel transport from points p to q does not depend on the chosen path. Then, $R(X, Y)Z = 0$ for all $X, Y, Z \in \Gamma(TM)$.*

Lemma 5.28. *Let $f : A \to M$ be a parametrized surface and let $V = V(s, t)$ be a vector field along the parametrized surface f. Then,*

$$\frac{D}{\partial t}\frac{D}{\partial s}V - \frac{D}{\partial s}\frac{D}{\partial t}V = R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)V$$