Introduction to Relativistic Fluids

Abstract

These notes follow a lecture I gave at the Vanderbilt Graduate Student Seminar on November 10, 2025. In these notes, I will outline the basic propositions of *classical fluids*. Then, I introduce relativistic inviscid fluids: the relativistic Euler equations. There is special attention paid to the free-boundary situation with the physical vacuum condition. Finally, I consider the linearized physical vacuum boundary problem and show the energy estimate and existence of solutions.

1 Classical Fluid Dynamics

A fluid is defined as a liquid or a gas. Water and air are both examples of a fluid. Fluid dynamics studies how different fluids behave in various regimes. The field spans across various disciplines: engineers, physicists, and mathematicians all have interests in the subject. The interests of each group vary rather dramatically, however. As mathematicians, we are firstly interested in the equations of motion of fluid regimes, which manifest as partial differential equations. We now ask the most basic question: when do solutions to the equations of motion for a fluid exist? Before studying fluids in a relativistic regime, we briefly review what are known as *classic fluids*. By this, we mean fluids that arise from Newtonian mechanics.

We now describe the basic setup of the classical fluid problem. In general, we work over a manifold M; in this case, consider the manifold as $\mathbb{R}^3$. Let $\Omega \subset \mathbb{R}^3$. We will use the coordinates (x_1, x_2, x_3) to describe the physical space of the fluid. We also will use the variable t to represent time in the evolution equations. We need to define some quantities that are used to describe the motion of a fluid.

Definition 1.1. We describe the following **macroscopic quantities**. The fluid's **velocity** is given by

$$u : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}^3.$$

Note that u is a vector field that has three components which we denote as $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$. The fluid's **density** is given by

$$\varrho : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}.$$

The fluid's **pressure** is given by

$$p : \mathbb{R}^+ \times \mathbb{R}^3 \to \mathbb{R}$$

1.1 The Equations of Motion

The equations of motion arise from conservation laws of classical physics. Specifically, they arise from the conservation of mass and conservation of momentum. Mathematically, these are given by

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho u) = 0 \tag{1}$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \frac{1}{\varrho}\nabla p - \nu \Delta u + \left(\frac{1}{3}\nu + \xi\right)\nabla(\nabla \cdot u) = 0 \tag{2}$$

where $\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ is the vector field nabla, ν is the kinematic viscosity, and ξ is the bulk kinematic viscosity of the fluid. Think of this as a measure of how fluid particles stick to the surfaces they touch. These equations are known as the **Navier-Stokes Equations**. Equation 1 is known as the conservation of mass or continuity equation. Equation 2 is known as the conservation of momentum equation. There are two simplifications that are often made to reduce the Navier-Stokes equations. The first is a restriction on the variable ϱ. There are some fluid regimes in which the fluid's density can be treated as constant. Think of a glass of water. Throughout the glass, the change in density is negligible. We call this type of fluid **incompressible**. If instead the density is not constant, we call this type of fluid as **compressible**. Assuming that $\varrho = c \in \mathbb{R}^+$, equation 1 becomes $\nabla \cdot u = 0$. In other words, the incompressibility condition yields that u is a divergence free vector field. We can therefore write the **incompressible Navier-Stokes equations** as

$$\nabla \cdot u = 0 \tag{3}$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p - \nu \Delta u = 0. \tag{4}$$

The second simplification we make pertains to the viscosity ν. For regimes in which the friction forces can be neglected, like a vehicle with low air resistance in space, the fluid is called **inviscid**. Such a fluid is also called **perfect**. In the classical sense, a perfect fluid is described the **Euler equations** and given by

$$\nabla \cdot u = 0 \tag{5}$$

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0. \tag{6}$$

1.2 The Material Derivative

We now motivate the idea of the **material derivative**. Consider a scalar field $f(t, x)$ and a velocity $u(t, x)$. We calculate the flow of the vector field u via

$$\frac{dx}{dt} = u(t, x).$$

Taking the time derivative of f, we have

$$\begin{aligned}\frac{d}{dt} f(t, x) &= \frac{\partial}{\partial t} f(t, x) + \nabla f \cdot \frac{dx}{dt} \\ &= \frac{\partial}{\partial t} f(t, x) + u \cdot \nabla f.\end{aligned}$$

We define the following quantity as the material derivative with respect to a velocity field u:

$$D_t \coloneqq \frac{\partial}{\partial t} + u \cdot \nabla. \tag{7}$$

We should think of the material derivative as following the material particle in the flow. In other words, the time derivative measures the local change over time while the material derivative measures the change as a particle moves in a vector field.

We can use the material derivative to rewrite the incompressible Euler equations as

$$\nabla \cdot u = 0 \tag{8}$$
$$D_t u + \nabla p = 0 \tag{9}$$

1.3 When does the classical fluid model fail?

The Navier-Stokes equations describes above arise from Newtonian mechanics. However, there are certain situations in which the effects of relativity cannot be neglected by the Newtonian model. Such examples include fluids traveling at speeds close to the speed of light or fluid's influenced by large gravitational fields. It is the latter that motivates the introduction of a **relativistic fluid** in the forthcoming sections. Consider a star with a large gravitational field, such a neutron star. This will be the main motivation.

2 Introduction to the Relativistic Euler Equations

We now derive the relativistic Euler equations in Minkowski space. First, we take a short detour into what we mean by special relativity.

2.1 The idea of relativity

We first give a general overview of what relativity means. There are two theories of relativity: special relativity and general relativity. For the focus of this talk, we respect ourselves to the study of special relativity. In the theory of special relativity, the speed of light is a constant and the same for any body moving in space. Nothing can go faster than the speed of light. The rest of the theory can be thought as the consequences of this fact. We will not dwell on this topic anymore as it not pertinent to the forthcoming mathematics.

2.2 Objects of Special Relativistic Fluids

We are interested in studying a prefect fluid in the background of special relativity. In the forthcoming discussion we assume we are working on a smooth manifold M with a Lorentzian metric g. We first introduce the energy-momentum tensor for a perfect fluid.

Definition 2.1. The **energy momentum tensor** for a perfect fluid is given by the following symmetric two-tensor on a manifold M

$$\mathcal{T}_{\alpha\beta} = (p + \varrho)u_\alpha u_\beta + p g_{\alpha\beta}, \tag{10}$$

where $\varrho = \varrho(t, x) : M \to \mathbb{R}$ is the fluid's **energy density**, $p : M \to \mathbb{R}$ is the fluid's **pressure**, $u = u(t, x) : M \to TM$ is the fluid's **four-velocity**, and $g = g(t, x)$ is a Lorentzian metric on M.

Before continuing, we need to address some notation that will be used throughout these notes. We work with coordinates of the form $\{x^\alpha\}_{\alpha=0}^3$ where one thinks of x^0 as the time variable and $\{x^1, x^2, x^3\}$ as the spatial variables. In general, Latin indices range from 1 to

n and Greek indices range from 0 to n. If we simply write x, it is usually meant to indicate the spatial components of x^α.

Definition 2.2. We define the **Baryon current** as the following vectorfield $\mathcal{J} : M \to TM$ given by

$$\mathcal{J}_\alpha = n u_\alpha, \tag{11}$$

where $n = n(t, x)$ is called the **Baryonic density**.

Definition 2.3. The pressure, energy density, and Baryonic density are related through an **equation of state** of the form

$$p = p(\varrho, n). \tag{12}$$

If the pressure is a function of only the energy density, we say that the fluid is **barotropic**. An example that we will use frequently is that of a polytrope. We define the equation of state for a polytope as

$$p(\varrho) = \varrho^{k+1} \qquad k > 0 \tag{13}$$

If the equation of state is a function of more than the energy density, we call it **non-barotropic**.

2.3 The relativistic Euler equations

Before we state the equations of motion, we have to introduce an important property of the velocity. Note that this is a purely physical statement. The velocity is subject to the following normalizing constraint equation:

$$g_{\alpha\beta} u^\alpha u^\beta = -1. \tag{14}$$

This property will be used over and over again in the forthcoming calculations. The assumption on the square norm of the four-velocity can be understood as follows. Recall that in relativity, observers are defined by their (timelike) world-line up to reparametrizations. More precisely, the norm of a tangent vector to the world-line has no physical meaning if the parameter is not specified. Thus, we can choose to normalize the observer's velocity to -1. In the case of a fluid, we can identify the flow lines of u with the world-line of observers traveling with the fluid particles. The normalization gives that u is timelike, so fluid particles do not travel faster than or at the speed of light. It also gives rise to the following important identity. Taking the covariant derivative of the above equation provides

$$u^\alpha \nabla_\beta u_\alpha = 0, \tag{15}$$

called the fluid's **acceleration** defined by

$$a^\alpha := u^\mu \nabla_\mu u^\alpha \tag{16}$$

The constraint equation also allows us to define a fluid's **local rest frame (LFR)** which is an orthonormal frame $\{e_\alpha\}_{\alpha=0}^3$ such that $e_0 = u$.

We can now define the relativistic Euler equations given in the following definition.

Definition 2.4. The **relativistic Euler equations** are given by

$$\nabla_\alpha \mathcal{T}^\alpha_\beta = 0 \tag{17a}$$
$$\nabla_\alpha \mathcal{J}^\alpha = 0 \tag{17b}$$
$$g_{\alpha\beta} u^\alpha u^\beta = -1 \tag{17c}$$
$$p = p(\varrho, n). \tag{17d}$$

Equation (17a) corresponds to conservation of energy and momentum and energy (17b) corresponds to conservation of Baryonic charge.

Remark. From physical considerations, we often require that $\varrho \geq 0$, $p \geq 0$, and $n \geq 0$. In fact, we will consider the case of zero pressure and density further in these notes.

We now discuss an alternative way to write the relativistic Euler equations.

Definition 2.5. Define the following two-tensor

$$\Pi_{\alpha\beta} \coloneqq g_{\alpha\beta} + u_\alpha u_\beta \tag{18}$$

which corresponds to projection onto the space orthogonal to u.

It is easy to see from the constraint equation that for a vector u

$$\Pi_{\alpha\beta} u^\beta = g_{\alpha\beta} u^\beta + u_\alpha \underbrace{u_\beta u^\beta}_{=0} = u_\alpha - u_\alpha = 0.$$

Furthermore, if v is orthogonal to u, we have

$$\Pi_{\alpha\beta} v^\beta = g_{\alpha\beta} v^\beta + u_\alpha u_\beta v^\beta = v_\beta.$$

To derive a more standard looking set of equations given by equation 17a, we decompose in the directions parallel and orthogonal to u:

$$\begin{aligned} \nabla_\mu T^\mu_\nu &= \nabla_\mu \left[(p+\varrho) u^\mu u_\nu + p g^\mu_\nu \right] \\ &= u^\mu \nabla_\mu (p+\varrho) u_\nu + (p+\varrho) \nabla_\mu u^\mu u_\nu + (p+\varrho) u^\mu \nabla_\mu u_\nu + \nabla_\nu \varrho \end{aligned}$$

If we multiply above by u^ν, we arrive at

$$\begin{aligned} u^\nu \nabla_\mu T^\mu_\nu &= u^\mu \nabla_\mu (p+\varrho) \underbrace{u_\nu u^\nu}_{-1} + (p+\varrho) \nabla_\mu u^\mu \underbrace{u_\nu u^\nu}_{=-1} + (p+\varrho) u^\mu \underbrace{u^\nu \nabla_\mu u_\nu}_{=0} + u^\nu \nabla_\nu \varrho \\ &= u_\mu \nabla_\mu \varrho + (p+\varrho) \nabla_\mu u^\mu \end{aligned}$$

which yields the *energy equation*

$$\boxed{u_\mu \nabla_\mu \varrho + (p+\varrho) \nabla_\mu u^\mu = 0} \tag{19}$$

Secondly, apply $\Pi^{\gamma\nu}$ to $\nabla_\mu T^\mu_\nu$:

$$\begin{aligned}
\Pi^{\gamma\nu}\nabla_\mu T^\mu_\nu &= u^\mu(p+\varrho)\underbrace{\Pi^{\gamma\nu}u_\nu}_{=0} + (p+\varrho)\nabla_\mu u^\mu \underbrace{\Pi^{\gamma\nu}u_\nu}_{=0} + (p+\varrho)\Pi^{\gamma\nu}u^\mu\nabla_\mu u_\nu + \Pi^{\gamma\nu}\nabla_\nu p \\
&= (p+\varrho)u^\mu\Big[\underbrace{g^{\gamma\nu}\nabla_\mu u_\nu}_{=\nabla_\mu u^\mu} + u^\gamma \underbrace{u^\nu\nabla_\mu u_\nu}_{=0}\Big] + \Pi^{\gamma\nu}\nabla_\nu p \\
&= (p+\varrho)u^\mu\nabla_\mu u^\nu + \Pi^{\nu\mu}\nabla_\mu p
\end{aligned}$$

which yields the *momentum equation*

$$\boxed{(p+\varrho)u^\mu\nabla_\mu u^\nu + \Pi^{\nu\mu}\nabla_\mu p = 0} \tag{20}$$

Therefore, we can write the relativistic Euler equations as

$$\begin{aligned}
u^\mu\partial_\mu\varrho + (p+\varrho)\nabla_\mu u^\mu &= 0 && (21)\\
(p+\varrho)u^\mu\nabla_\mu u^\nu + \Pi^{\nu\mu}\nabla_\mu p &= 0 && (22)\\
u^\mu\nabla_\mu n + n\nabla_\mu u^\mu &= 0 && (23)\\
g_{\mu\nu}u^\mu u^\nu &= -1 && (24)\\
p &= p(\varrho, n). && (25)
\end{aligned}$$

2.4 Thermodynamics

We introduce the following quantities:

Definition 2.6.

1. The **internal energy**, E given by

$$\varrho = n(1+E). \tag{26}$$

2. The **enthalpy**, h given by

$$h = \frac{p+\varrho}{n} \qquad n > 0. \tag{27}$$

3. The **specific entropy**, called s, and **temperature** , called θ. These quantities are related by the following first law of thermodynamics.

Proposition 2.7. *(The First Law of Thermodynamics)*

$$dp = ndh - n\theta ds, \tag{28}$$

where d is the exterior derivative in spacetime.

Using the above relation, we can derive an equation of motion for the specific entropy, s. If we assume the physically natural stipulations that $\theta > 0$ and $n > 0$, we find that

$$u^\mu\nabla_\mu s = 0. \tag{29}$$

Physically, this relation yields that the fluid is locally adiabatic, meaning that the entropy is constant along the fluid's flow lines.

3 Free Boundary Relativistic Euler

We now consider the relativistic Euler equations in a fixed background. For simplicity, we consider the background of the Minkowski metric. We begin with a discussion about free-boundary problems.

3.1 Free Boundary Problem

Relativistic fluids are often modeled as a free-boundary. A free boundary problem means that the boundary evolves with the problem. Think of a wave moving in the ocean; as the time parameter in the problem increases, the boundary of the domain changes. This is an example of a free boundary.

We are going to consider whose domain is not fixed, but rather moves with the motion of the fluid. We call this type of fluid as a **free-boundary fluid**. In the context of relativistic fluids, this could model a relativistic star in a vacuum.

At a time t, let Ω_t be the region occupied by the fluid. We can then define the space occupied by the fluid at all times $0 \leq t \leq T$ for some $T > 0$ as

$$\Omega \coloneqq \bigcup_{0 \leq t \leq T} \Omega_t \times \{t\}. \tag{30}$$

We call this as the **moving domain**. We are interested in the boundary of the fluid that we call the **free-boundary**. It is defined as

$$\Gamma \coloneqq \bigcup_{0 \leq t \leq T} \Gamma_t \times \{t\}. \tag{31}$$

We are interested in the **free-boundary relativistic Euler equations** in the domain Ω. Outside of the fluid, we assume that there is a vacuum. This is why we use a relativistic star as our main example.

3.2 Boundary Conditions

The variables of interest in this problem are the fluid velocity, density, and pressure. Naturally, we need to prescribe boundary conditions for each of the above variables. The following conditions come largely as physical requirements rather than mathematical. We impose the following conditions on the pressure and velocity at the boundary:

$$p|_\Gamma = 0 \tag{32}$$

and

$$u \in T\Gamma \tag{33}$$

where $T\Gamma$ is the tangent bundle of Γ. The first condition can be intuitively understood given that outside of the fluid is a vacuum. The velocity condition says that Γ_t is advected by the fluid. That is, Γ_t moves at a speed of the normal component of the fluid at the boundary.

Moreover, we assume that the fluid is **barotropic**. This means that the fluid's pressure is a function only of its density. That is

$$p = p(\varrho). \tag{34}$$

Understanding this assumption, we have two cases for the density at the free-boundary:

- $\varrho|_\Gamma > 0$, which is the case of a **liquid**.
- $\varrho|_\Gamma = 0$, which is the case of a **gas**.

We will restrict ourselves to the second case, that of a gas. Note that these two cases present very different mathematical problems. In our case of a gas, we see that the equations degenerate at the boundary. That comes from the quantity $(p+\varrho)$ vanishing at the boundary given the above conditions. In the case of a gas, we call the free boundary as the **vacuum boundary**. We also have another description of the domain Ω as

$$\Omega = \{(\tau, x) \mid \tau = t, \varrho(t, x) > 0\}. \tag{35}$$

Finally, given that the outside of the fluid is a vacuum, sound waves cannot exist. Hence, we impose that

$$c_s^2|_\Gamma = 0. \tag{36}$$

It turns out that the decay rate of the speed of sound squared near the boundary plays an important role in this problem. We will show that there is only one physical decay rate for the speed of sound squared near the boundary: one that is linear. That is

$$c_s^2(t, x) \approx \operatorname{dist}(x, \Gamma_t), \tag{37}$$

for $x \in \Omega_t$ near Γ_t.

3.3 Diagonalized Equations

To end this section on the free-boundary relativistic Euler equations, we make some substitutions which will help in analysis in the future. Specifically, we want to diagonalize equations about the material derivative. We now seek to choose good dynamical variables that are tailored to the characteristics of the Euler flow all the way to the moving boundary. Here, the sound characteristics will vanish due to the vacuum boundary condition. Our choice of good variables will

1. better diagonalize the system with respect to the material derivative,

2. be associated with truly relativistic properties of the vorticity, and

3. leads to good weights that are all for the control of the behavior of the fluid variables when one approaches the boundary.

Property 1 is important because it is tied with both the wave and transport character of the flow in that (a) the diagonalized equations lead to good second order equations that capture the propagation of the flow and (b) it will allow for a good transport structure that will allow us to implement a time discretization for the construction of regular solutions. Property 2 ensures a good coupling between the wave-part and the transport-part of the system. Finally, property 3 will lead to the correct functional framework needed to close the estimates.

We will denote the good variables by (r, v). First, we want v to be a rescaled version of the velocity written as

$$v^\nu = f(\varrho)u^\nu \tag{38}$$

where f is given by

$$f(\varrho) \coloneqq \exp\left(\int \frac{c_s^2}{p(\varrho)+\varrho} d\varrho\right). \tag{39}$$

First, we consider a general barotropic equation of state. Computing the derivative of v^μ yields

$$\partial_\mu v^\nu = f'(\varrho)\partial_\mu \varrho u^\nu + f(\varrho)\partial_\mu u^\nu.$$

We can implicitly substitute into equation (22) and find

$$\frac{p+\varrho}{f} u^\mu \partial_\mu v^\alpha + h^2 m^{\alpha\mu}\partial_\mu \varrho + \left(-\frac{f'}{f}(p+\varrho) + h^2\right) u^\alpha u^\mu \partial_\mu \varrho = 0.$$

Now our choice for f comes to light. Using equation (39), we see that the coefficients on the last term above vanish. This satisfies the above requirements because the resulting equation is diagonal with respect to the material derivative and given by

$$D_t v^\nu + \frac{c_s^2 f^2}{(p+\varrho)v^0} m^{\nu\mu}\partial_\mu \varrho = 0. \tag{40}$$

We can use the velocity constraint condition to solve for v^0, which gives

$$v^0 = \sqrt{f^2 + |v|^2}, \qquad |v|^2 \coloneqq v^i v_i. \tag{41}$$

Note that in solving for v^0, in order for v^0 to be a future-pointing vector field, we choose the positive square root. This gives a diagonal equations because we can eliminate all the time derivatives given we can explicitly solve for v^0 given the already calculated quantities.

Next, we would like to diagonalize equation (21) using the new variable v. First, we solve for $\partial_t v^\alpha$ by setting $\alpha = 0$ in equation (40) to obtain

$$\begin{aligned}
\partial_t v^0 &= \frac{h^2 f^2}{(p+\varrho)v^0}\partial_t \varrho - \frac{v^i}{v^0}\partial_i v^0 \\
&= \frac{h^2 f^2}{(p+\varrho)v^0}\partial_t \varrho - \frac{f f'}{(v^0)^2} v^i \partial_i \varrho - \frac{v^i v^j}{(v^0)^2}\partial_i v_j,
\end{aligned}$$

where we use equation (41) to compute $\partial_i v^0$. We can now solve for $\partial_\mu v^\mu$ using the above relation to find

$$\partial_\mu v^\mu = \frac{h^2 f^2}{(p+\varrho)v^0}\partial_t \varrho - \frac{f f'}{(v^0)^2} v^i \partial_i \varrho - \left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right)\partial_i v_j$$

where δ is the Euclidean metric. We now express $\partial_\mu u^\mu$ in terms of $\partial_\mu v^\mu$. After some algebra, we can write equation (**??**) as

$$D_t\varrho + \frac{p+\varrho}{a_0 v^0}\left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right)\partial_i v_j - h^2 \frac{2f^2}{a_0 (v^0)^3} v^i \partial_i \varrho = 0, \tag{42}$$

where

$$a_0 := 1 - h^2 \frac{|v|^2}{(v^0)^2}. \tag{43}$$

We have equations (42) and (40) that are valid for a general barotropic equation of state. Moving forward, we will make a specific choice of equation of state. We will assume that the equation of state is a polytropic gas. This is given by

$$p(\varrho) = \varrho^{\kappa+1} \tag{44}$$

where $\kappa > 1$. Making this choice will allow us to define the second good variable. We can calculate the speed of sound squared to be $h^2 = (\kappa+1)\varrho^\kappa$ and the function f becomes $f(\varrho) = (1+\varrho^\kappa)^{1+\frac{1}{\kappa}}$. It is the case that it is better to adopt the sound speed squared as a primary variable instead of ϱ as it plays the role of the correct weight in the forthcoming energy functionals. We define the second component of the good variables by

$$r := \frac{1+\kappa}{\kappa}\varrho^\kappa. \tag{45}$$

Making the substitutions into equations (42) and (40) yields the following good variable formulation of the relativistic Euler system.

$$D_t r + rG^{ij}\partial_i v_j + r a_1 v^i \partial_i r = 0 \tag{46a}$$

$$D_t v_i + a_2 \partial_i r = 0 \tag{46b}$$

where we have defined

$$G^{ij} := \frac{\kappa \tilde{r}}{a_0 v^0}\left(\delta^{ij} - \frac{v^i v^j}{(v^0)^2}\right), \quad \tilde{r} := 1 + \frac{\kappa r}{\kappa+1}$$

and the coefficients a_0, a_1, and a_2 are given by

$$a_0 := 1 - \kappa r \frac{|v|^2}{(v^0)^2}, \quad a_1 := -\frac{2\kappa \tilde{r}^{2+\frac{2}{\kappa}}}{(v^0)^3 a_0}, \quad a_2 := \frac{\tilde{r}^{1+\frac{2}{\kappa}}}{v^0}.$$

Equations (46) are the desired form of the system as they are diagonalizable with respect to the material derivative. These equations will be the main concern of the forthcoming analysis. We will consider only the spatial components v^i as variables and v^0 given by

$$v^0 = \sqrt{\tilde{r}^{2+\frac{2}{\kappa}} + |v|^2}. \tag{47}$$

The specific form of the coefficients a_i for $i \in \{0,1,2\}$ is not crucial to the arguments. The takeaway from these constants is that they are smooth functions of r and v and $a_0, a_2 > 0$.

The operator $G^{ij}\partial_i(\cdot)_j$ can be viewed as a divergence-type operator. This divergence structure is related to the fact that equations (46) express the wave-like behavior of r and the divergence part of v. The symmetric and positive definite matrix $c_s^2 H^{ij}$ is closely related to the inverse of the acoustical metric. This is manifested in their agreement at the leading order near the boundary.

Another consequence of equations (46) is that they have the correct balance of powers of r to allow estimates all the way to the free boundary. The r factor in the divergence of v is related to the propagation of sound in the fluid, whereas the r factor in the last term of equation (46a) will allow us to treat it as a perturbation term in the forthcoming elliptic estimates.

4 The Linearized Equation

We end this introduction to relativistic fluids with a section on the linearized equations of equations 46. In short, we want to show that these equations are **well-posed**.

Definition 4.1. For a system of PDEs, we say that the system is **well-posed** if the following three conditions hold:

1. For a given $T > 0$, there exists a solution to the system.
2. For the above solution, it is unique.
3. The solution behaves nicely if the initial conditions are altered.

Before we can prove that the linearized system is well posed, we must define the spaces in which we work. Traditionally, solutions to PDEs live in **Sobolev spaces**. Think of these as spaces that mimic the Lebesgue spaces L^p, but they give information about 'derivatives' of the elements. We need to consider a slightly modified type of Sobolev space. The reason has to do with the behavior of this problem near the boundary. It is clear that the equations of motion degenerate at the boundary. This has to do with the vacuum boundary condition discussed above.

4.1 Weighted Sobolev spaces

Definition 4.2. Define the space $L^2(\Omega_t)[r]$ as the L^2 weighted spaces over Ω_t whose norm is given by

$$\|\cdot\|^2_{L^2(\Omega_t)[r]} \coloneqq \int_{\Omega_t} r|\cdot|^2 dx \tag{48}$$

We now define the weighted Sobolev spaces.

Definition 4.3. For an integer $j \geq 0$ and $\sigma > -1/2$, define the space $H^{j,\sigma}(\Omega_t)$ to be the space of distributions in Ω_t whose norm given by

$$\|\cdot\|^2_{H^{j,\sigma}(\Omega_t)} \coloneqq \sum_{|\alpha|\leq j} \|r^\sigma \partial^\alpha \cdot\|^2_{L^2(\Omega_t)} \tag{49}$$

is finite. Note that using interpolation, we can extend this definition to all real $s \geq 0$, and hence define $H^{s,\sigma}(\Omega_t)$.

We now present some important properties about weighted Sobolev spaces that will be useful later.

Lemma 4.4. ***Weighted Embedding***. *For integers $N, k \geq 0$, and $\sigma \geq 0$, we have*

$$H^{N+k,\sigma+k} \subset H^{N,\sigma}. \tag{50}$$

Lemma 4.5. ***Embedding relative to weights***. *For integers $N, \tau \geq 0$, and $\sigma \geq 0$, we have*

$$H^{N,\sigma} \subset H^{N,\sigma+\tau}. \tag{51}$$

Lemma 4.6. ***Embedding into C^k spaces*** *For integers $N, k \geq 0$ and $\sigma \geq 0$, we have*

$$H^{N,\sigma} \subset C^k \tag{52}$$

for $0 \leq k < N - \sigma - \frac{1}{2}$.

In the coming estimate, we are going to need to take time derivatives of integrals over the free boundary. This procedure requires some delicate handling. We present the following result on how to do this.

Theorem 4.7. ***(Reynolds Transport Theorem)*** *For a time-dependent domain Ω_t, we have the following differentiation formula*

$$\frac{d}{dt}\int_{\Omega_t} f dx = \int_{\Omega_t} D_t f dx + \int_{\Omega_t} f \partial_i \left(\frac{u^i}{u^0}\right) dx. \tag{53}$$

We now present the linearized relativistic Euler system. When working with PDEs, it is common to consider the linearized case before moving to the nonlinear case. In fact, we can use the results of the linearized system as a tool to aid in the nonlinear analysis. We introduce the variable s to be the variable associated to the linearization of r and w to be the variable associated with the linearization of v. A computation gives the following linearization of (46) as

$$D_t s + \frac{1}{\kappa} H^{ij} \partial_i r w_j + r H^{ij} \partial_i w_j + r a_1 v^i \partial_i s = f \tag{54a}$$

$$D_t w_i + a_2 \partial_i s = h_i \tag{54b}$$

where f and h are of the form

$$f = V_1 s + r W_1 w \tag{55a}$$

$$h = V_1 s + W_2 w \tag{55b}$$

such that S_1, S_2, W_1, W_2 are linear in $\partial(r, v)$ with coefficients that are smooth functions of (r, v). The functions f and h are treated as the error terms. These will be controlled in a later section. An important consequence of the linearized system is that it does not obtain nor require any boundary conditions on the free boundary Γ_t. This is the case because the one-parameter family of solutions is not required to have the same domain, as it would be if one were working in Lagrangian coordinates.

Finally, we provide explicit expressions for the potentials $V_{1,2}$ and $W_{1,2}$. However, these do not play a significant role in the rest of this work. We have

$$V_1 = \frac{v^j}{(v^0)^3}\tilde{r}^{1+\frac{2}{\kappa}}\partial_j r - H^{ij}\partial_i v_j - r\frac{\partial H^{ij}}{\partial r}\partial_i v_j, \tag{56a}$$

$$W_2^l = -\frac{\partial H^{ij}}{\partial v^l}\partial_i v_j - ra_3 H^{il}\partial_i r, \tag{56b}$$

$$V_{2,i} = -\frac{v^j}{(v^0)^3}\tilde{r}^{1+\frac{2}{\kappa}}\partial_j v_j + \frac{\partial a_2}{\partial r}\partial_i r, \tag{56c}$$

$$(W_2)_i^l = -\frac{a_0}{\kappa\tilde{r}}H^{jl}\partial_j v_j + \frac{\partial a_2}{\partial v^l}\partial_i r, \tag{56d}$$

where a_3 is a smooth function of (r, v) given by

$$\frac{a_0}{\kappa\tilde{r}} - \frac{1}{\kappa} = ra_3, \qquad a_3 = -\frac{1}{\tilde{r}}\left(\frac{1}{2} + \frac{|v|^2}{(v^0)^3}\right). \tag{57}$$

4.2 The Linearized Energy Estimate

To derive an energy estimate for the linearized equation, we need to define the appropriate energy functional. Consider the following definition.

Definition 4.8. For the purpose of the forthcoming analysis, view the time-dependent space as a Hilbert space whose squared norm is now defined. Define the energy functional for the linearized equations as

$$E_{\text{lin}}(s, w) := \|(s, w)\|^2 = \int_{D_t} r^{\frac{1-\kappa}{\kappa}}(s^2 + a_2^{-1}rH^{ij}w_i w_j)dx. \tag{58}$$

We will use this space for the linearized equations and its adjoint. We now present the main result for the linearized equations.

Proposition 4.9. *Let (r, v) be a solution to system (54). Assume that both r and v are Lipschitz continuous and that r vanishes simply on the free boundary. Then, the following estimate holds for solutions (s, w) to equations (54):*

$$\left|\frac{d}{dt}\|(s, w)\|^2\right| \lesssim B\|(s, w)\|^2. \tag{59}$$

Proof. We first consider the case of $\kappa = 1$. Multiply equation (54a) by s and contract equation (54b) by $a_2^{-1}rH^{ij}w_j$. We have that

$$\frac{1}{2}D_t(s^2) + sH^{ij}\partial_i r w_j + rsH^{ij}\partial_i w_j + sra_1 v^i\partial_i s = sf$$

$$\frac{1}{2}a_2^{-1}rH^{ij}D_t(w_i w_j) + rH^{ij}w_j\partial_i s = a_2^{-1}rH^{ij}w_j h_i$$

and summing the two equations yields

$$\frac{1}{2}D_t(s^2) + \frac{1}{2}a_2^{-1}rH^{ij}D_t(w_iw_j) + sH^{ij}\partial_i rw_j + rsH^{ij}\partial_i w_j + sra_1v^i\partial_i s + rH^{ij}w_j\partial_i s$$
$$= sf + a_2^{-1}rH^{ij}w_jh_i.$$

To derive the energy estimate for this case, we integrate over D_t and we arrive at

$$\begin{aligned}\frac{1}{2}\int_{D_t} D_t(s^2) + a_2^{-1}rH^{ij}D_t(w_iw_j)dx + \int_{D_t} sH^{ij}\partial_i rw_jdx + \int_{D_t} rsH^{ij}\partial_i w_jdx \\ + \int_{D_t} sra_1v^i\partial_i sdx + \int_{D_t} rH^{ij}w_j\partial_i sdx = \int_{D_t} sf + a_2^{-1}rH^{ij}w_jh_idx.\end{aligned} \tag{60}$$

We first use the Reynolds transport theorem (53) on the first integral on the left hand side of (60) to find

$$\frac{1}{2}\int_{D_t} D_t(s^2)dx = \frac{1}{2}\frac{d}{dt}\int_{D_t} s^2dx - \frac{1}{2}\int_{D_t} s^2\partial_i\left(\frac{v^i}{v^0}\right)dx \tag{61}$$

We can easily bound the second integral above by part of the energy as

$$\left|\int_{D_t} s^2\partial_i\left(\frac{v^i}{v^0}\right)dx\right| \le \left\|\partial_i\left(\frac{v^i}{v^0}\right)\right\|_{L^\infty(D_t)}\int_{D_t} s^2dx.$$

Similarly,

$$\begin{aligned}\frac{1}{2}\int_{D_t} a_2^{-1}rH^{ij}D_t(w_iw_j)dx &= \frac{1}{2}\int_{D_t} D_t\left(a_2^{-1}rH^{ij}w_iw_j\right)dx - \frac{1}{2}\int_{D_t} D_t\left(a_2^{-1}rH^{ij}\right)w_iw_jdx \\ &= \frac{1}{2}\frac{d}{dt}\int_{D_t} a_2^{-1}rH^{ij}w_iw_jdx - \frac{1}{2}\int_{D_t} a_2^{-1}rH^{ij}w_iw_j\partial_i\left(\frac{v^i}{v^0}\right)dx \\ &\quad - \frac{1}{2}\int_{D_t} D_t\left(a_2^{-1}rH^{ij}\right)w_iw_jdx\end{aligned} \tag{62}$$

As before, we can easily bound the sound integral above as part of the energy as

$$\left|\int_{D_t} a_2^{-1}rH^{ij}w_iw_j\partial_i\left(\frac{v^i}{v^0}\right)dx\right| \le \left\|\partial_i\left(\frac{v^i}{v^0}\right)\right\|_{L^\infty}\int_{D_t} a_2^{-1}rH^{ij}w_iw_jdx$$

Combining this bound with the previous estimate yields the following energy bound

$$\left|\int_{D_t}\left(s^2 + a_2^{-1}rH^{ij}w_iw_j\right)\partial_i\left(\frac{v^i}{v^0}\right)dx\right| \lesssim \|(s,w)\|^2$$

Next, we consider the third integral above and split it up into

$$\frac{1}{2}\int_{D_t} D_t\left(a_2^{-1}rH^{ij}\right)w_iw_jdx = \frac{1}{2}\int_{D_t} rD_t\left(a_2^{-1}H^{ij}\right)w_iw_jdx + \frac{1}{2}\int_{D_t} D_t(r)a_2^{-1}rH^{ij}w_iw_jdx. \tag{63}$$

As before, the first integral can be controlled by the energy as

$$\frac{1}{2}\int_{D_t} rD_t\left(a_2^{-1}H^{ij}\right)w_iw_jdx \lesssim \|(s,w)\|^2.$$

The second integral seems to be missing the correct power of r in order to be controlled by the energy. Indeed, the vacuum boundary condition provides $r(t,x) \approx \operatorname{dist}(x,_t)$, which yields $\overline{\partial}r = O(1)$ so that

$$\int_{D_t} \overline{\partial}r|w|^2dx \approx \int_{D_t}|w|^2dx \not\lesssim \int_{D_t} r|w|^2dx.$$

However, we note that the derivative that falls upon r is *not* a simple spatial derivative; it is a material derivative. This fact is crucial. Using equation (46a), we have

$$D_tr = r\overline{\partial}(r,v) \tag{64}$$

and hence we require the correct weight for the energy estimates. Using this miracle, we can control the second integral as

Next, we consider the fourth integral on the left-hand side of equation (60). We find that

$$\begin{aligned}\left|\int_{D_t} sra_1v^i\partial_isdx\right| &= \left|\frac{1}{2}\int_{D_t} a_1rv^i\partial_i\left(s^2\right)dx\right|\\ &\le \int_{D_t}\left|\partial_i\left(a_1rsv^i\right)\right|s^2dx\\ &\lesssim \int_{D_t} s^2dx\\ &\lesssim \|(s,w)\|^2.\end{aligned}$$

The final three terms on the left-hand side of equation (60) can be written in divergence form. Using integration by parts, we can bound the divergence form by the energy in the following way

$$\begin{aligned}\left|\int_{D_t} sH^{ij}\partial_irw_jdx + \int_{D_t} rsH^{ij}\partial_iw_jdx + \int_{D_t} rH^{ij}w_j\partial_isdx\right| &\le \int_{D_t}\left|H^{ij}\partial_i(rsw_j)\right|dx\\ &= \left|\int_{D_t}\partial_i(H^{ij})(rsw_j)dx\right|\\ &\lesssim \|(s,w)\|^2\end{aligned}$$

Finally, we turn to the right-hand side of equation (60). The Cauchy-Schwartz inequality gives

$$\int_{D_t} sf + a_2^{-1}rH^{ij}w_jh_idx \le \|(f,g)\|\|(s,w)\|.$$

It remains to show that (f,g) are perturbative terms. We note the following bound on the potentials given by equations (56)

$$\|V_{1,2}\|_{L^\infty(D_t)} + \|W_{1,2}\|_{L^\infty(D_t)} \lesssim B. \tag{65}$$

Using this bound, it follows that

$$\begin{aligned}\|(f,g)\|^2 &= \int_{D_t} (f^2 + a_2 r H^{ij} g_i g_j)dx \\ &= \int_{D_t} ((V_1 s + rW_1 w_i)^2 + a_2 r H^{ij} g_i g_j)dx \\ &\lesssim B\|(s,w)\|^2\end{aligned}$$

Combining the bounds shown above, we obtain

$$\left|\frac{d}{dt}\|(s,w)\|^2\right| \lesssim B\|(s,w)\|^2 + \|(s,w)\|\|(f,g)\| \lesssim B\|(s,w)\|^2.$$

We now outline the result for a general κ. The only significant difference is that which we multiply or contract the equations by to yield the correct divergence form. Multiply equation 54a by $r^{\frac{1-\kappa}{\kappa}}s$ and contract equation 54b with $a_2^{-1}r^{\frac{1}{\kappa}}H^{ij}w_j$ to find that

$$\frac{1}{2}r^{\frac{1-\kappa}{\kappa}}D_t s^2 + \frac{1}{\kappa}r^{\frac{1-\kappa}{\kappa}}H^{ij}\partial_i r w_j s + r^{\frac{1}{\kappa}}H^{ij}\partial_i w_j s + \frac{1}{2}r^{\frac{1}{\kappa}}a_1 v^i \partial_i s^2 = fr^{\frac{1-\kappa}{\kappa}}s,$$

$$\frac{1}{2}a_2^{-1}r^{\frac{1}{\kappa}}G^{ij}D_t(w_i,w_j) + r^{\frac{1}{\kappa}}H^{ij}w_j\partial_i s = a_2^{-1}r^{\frac{1}{\kappa}}H^{ij}g_i w_j.$$

The next step is to add the above two equations together. In doing so, we can simplify by writing three of the terms in divergence form. That is

$$\begin{aligned}\frac{1}{\kappa}r^{\frac{1-\kappa}{\kappa}}H^{ij}\partial_i r w_j s + r^{\frac{1}{\kappa}}H^{ij}\partial_i w_j s + r^{\frac{1}{\kappa}}H^{ij}w_j\partial_i s &= \partial_i\left(r^{\frac{1}{\kappa}}\right)H^{ij}w_j s + r^{\frac{1}{\kappa}}H^{ij}\partial_i w_j s + r^{\frac{1}{\kappa}}H^{ij}w_j\partial_i s \\ &= H^{ij}\partial_i\left(r^{\frac{1}{\kappa}}w_j s\right).\end{aligned}$$

Using this observation, adding the two equations yields

$$\frac{1}{2}r^{\frac{1-\kappa}{\kappa}}D_t s^2 + \frac{1}{2}a_2^{-1}r^{\frac{1}{\kappa}}H^{ij}D_t(w_i w_j) + \frac{1}{2}r^{\frac{1}{\kappa}}a_1 v^i\partial_i s^2 + H^{ij}\partial_i\left(r^{\frac{1}{\kappa}}w_j s\right) = fr^{\frac{1-\kappa}{\kappa}}s + a_2^{-1}r^{\frac{1}{\kappa}}H^{ij}g_i w_j.$$

From here, the rest of the estimates are almost identical to the case of $\kappa = 1$. This concludes the proof of the energy estimate of the linearized equations. □

We now show local well-posedness of the linearized system.

Proposition 4.10. *Let (r,v) be a solution to system (46) and assume that r and v are Lipschitz continuous and that r vanishes simply on the free boundary. Then, the linearized equations (54) are well-posed in .*

Proof. We first compute the adjoint equation to (54) with respect to the duality relation defined by the inner product determined by norm (58). The terms f and g on the right-hand side of equation (54) are linear expressions in s and rw and in s and w, respectively, with $\partial(r,v)$ coefficients. Thus, the source terms in the adjoint equation have the same structure as the original equation. We can write the left hand side of equation (54) as

$$D_t\begin{pmatrix}s\\w\end{pmatrix} + A^i\partial_i\begin{pmatrix}s\\w\end{pmatrix} + B\begin{pmatrix}s\\w,\end{pmatrix}$$

where

$$A^i = \begin{bmatrix} a_1 r v^i & r H^{ij} \\ a_2^{il} & 0_{3\times 3} \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0_{1\times 1} & \frac{1}{\kappa} H^{ij} \\ 0_{3\times 1} & 0_{3\times 3} \end{bmatrix}$$

With respect to the inner product, the adjoint term corresponding to $A^i \partial_i$ is

$$\tilde{A}^i \partial_i = - \begin{bmatrix} a_1 r v^i & r H^{ij} \\ a_2^{il} & 0_{3\times 3} \end{bmatrix} \partial_i - \begin{bmatrix} 0_{1\times 1} & \frac{1}{\kappa} H^{ij} \partial_i r \\ \frac{1}{\kappa} r^{-1} a_2 \partial_l r & 0_{3\times 3} \end{bmatrix}$$

modulo terms that are nonlinear expressions in $\tilde{s}$ and $r\tilde{w}$ and in $\tilde{s}$ and $\tilde{w}$ (with $\partial(r, v)$ coefficients) in the first and second components, respectively, where $\tilde{s}$ and $\tilde{w}$ are elements of the dual. Similarly, the adjoint term corresponding to B is

$$\tilde{B} = \begin{bmatrix} 0_{1\times 1} & 0_{1\times 3} \\ \frac{1}{\kappa} r^{-1} a_2 \partial_l r & 0_{3\times 3}. \end{bmatrix}$$

Combining these expressions, we see that the bad term on the lower left corner of the second matrix in $\tilde{A}^i \partial_i$ cancels with the corresponding terms in $\tilde{B}$. Therefore, the adjoint problem is the same as the original one, modulo perturbative terms, and it therefore admits an energy estimate similar to the energy estimate for the linearized equations.

The forward energy estimate for the linearized equation and the backward in time energy estimate for the adjoint linearized equations yield uniqueness, respectively, existence of solutions for the linearized equation, as needed. This guarantees the well-posedness of the linearized system. □