Differential Topology Notes

Abstract

These notes are from Math 6120 taught in Fall 2025 by Professor Ioana Suvaina and prepared by Brian Morton.

Contents

1 Review of Point Set Topology

Before we dive into differential geometry, we will review some important aspects of point set topology. We begin with the definition of a topology.

Definition 1.1. A **topology** for a set X is a collection $\mathcal{T}$ of subsets of X such that:

1. The intersection of any two members of $\mathcal{T}$ is also in $\mathcal{T}$.
2. The union of any number of members of $\mathcal{T}$ is also in $\mathcal{T}$.
3. $X \in \mathcal{T}$
4. $\emptyset \in \mathcal{T}$

We call the pair $(X, \mathcal{T})$ a **topological space** and the elements of X are called **points**. The members of $\mathcal{T}$ are called **open subsets** of X.

Example 1.2. We provide two trivial examples of a topology on a set X.

1. The **discrete topology**: $\mathcal{T} = \{\text{All subsets of } X\}$.

2. The **indiscrete topology**: $\mathcal{T} = \{\emptyset, X\}$.

Definition 1.3. A **basis** for a topology for X is a collection $\mathcal{B} = \{B_i\}_{i \in I}$ of subsets of X such that

1. $\bigcup_i B_i = X$,

2. if $x \in B_i \cap B_j$, then there exists a $k \in I$ such that $x \in B_k \subset B_i \cap B_j$.

Definition 1.4. A **subbasis** for a topology $\mathcal{T}$ for a set X is a collection $\mathcal{A} = \{A_i\}_{i \in I}$ of subsets of X such that the family of finite intersections of members of $\mathcal{A}$ is a basis for $\mathcal{T}$.

Definition 1.5. A **neighborhood** of $x \in X$ is an open set of X containing x. A subset of X is open if and only if it is a neighborhood of each of its points. A subset is closed if its complement is open.

We now introduce some important definitions of properties of a space X.

Definition 1.6. Let X be a space.

1. We say that X is a T_0-space if, for each pair of distinct points $x, y \in X$, there is a neighborhood U_x of X such that $y \notin U_x$ or there is a neighborhood U_y of y such that $x \notin U_y$.

2. We say that X is a T_1-space if, for each pair of distinct points $x, y \in X$, there exist neighborhoods U_x and U_y of x and y, respectively, such that $x \notin U_y$ and $y \notin U_x$.

3. We say that X is a T_2-space if, for each pair of distinct points $x, y \in X$, there exist disjoint neighborhoods U_x of x and U_y of y. A T_2-space is also called a **Hausdorff space**.

4. We say that X is a T_3-space if, given any closed set $D \subset X$ and a point $x \in X - D$, there exist open sets U and V such that $U \cap V$, $D \subset U$, and $x \in V$. We say X is **regular** if it is T_1 and T_3.

5. A space X is T_4 if, for each pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$. We say that X is **normal** if it is T_1 and T_4.

2 Introduction to Differential Manifolds

Definition 2.1. A **topological manifold** of dimension n is a set M such that

1. M is a Hausdorff space

2. at each point in M, there is a neighborhood U with is homeomorphic to an open set in $\mathbb{R}^n$.

Definition 2.2. If the charts (U, φ) and (V, ψ) are such that $U \cap V = \emptyset$, the composite map

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$$

is called the **transition map** from φ to ψ, We say that two charts (U, φ) and (V, ψ) are C^∞ **comparable** if their compositions $\varphi \circ \phi^{-1}$ and $\phi \circ \varphi^{-1}$ are also C^∞.

Definition 2.3. A **differentiable structure** on a topological manifold M is given by a family $\mathcal{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in I}$ of coordinate neighborhoods such that

1. $\bigcup_\alpha U_\alpha = M$
2. for all $\alpha, \beta \in I$, the charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are C^∞ comparable
3. Any chart (V, ψ) comparable with $(U_\alpha, \varphi_\alpha)$ is itself in $\mathcal{U}$.

A C^∞ manifold is a topological manifold together with a C^∞ differentiable structure on the manifold. If $(M,\mathcal{U})$ satisfies properties 1 and 2, we call this an **atlas** on M. If $(M,\mathcal{U})$ satisfies properties 1, 2, and 3, we call this a **maximal atlas** on M.

Remark. Two atlases are comparable if their union forms an atlas.

Proposition 2.4. *Let M be a topological manifold. Then, the following hold:*

1. *Every smooth atlas $(M,\mathcal{U})$ is contained in a unique maximal smooth atlas.*
2. *Two atlases determine the same smooth if and only if their union is a smooth atlas.*

2.1 Examples

We now provide a set of examples of smooth manifolds.

1. The Cartesian plane $\mathbb{R}^n$ is a smooth manifold with atlas $\mathcal{U} = \{(\mathbb{R}^n, id)\}$.
2. The n-sphere defined by

$$S^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + ... + x_{n+1}^2 = 1\} \tag{1}$$

is a smooth manifold. We provide two examples of atlases. The first atlas has two charts given by stereographic projection. Define the north and south poles as

$$N = (1, 0, ..., 0) \qquad S = (-1, 0, ..., 0)$$

and the stereographic projection given by

$$\Pi_N : S^n n\{N\} \to \mathbb{R}^n$$

$$(x_1, ..., x_{n+1}0 \mapsto \left(\frac{x_2}{1 - x_1}, ..., \frac{x_n}{1 - x_1}\right)$$

3. Hyperboloid: The hyperboloid is defined as

$$H^n = \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 - \left(x_2^2 + .. + x_{n+1}^2\right) = 1\}. \tag{2}$$

The smooth structure is given by two charts. The first is given by

$$H_+ = \{(x_1, .., x_{n+1}) \in H^n \mid x_1 > 0\}$$

with the projection given by

$$\Pi_+ : H_+ \to$$

Proposition 2.5. *We have the following proposition pertaining to products of manifolds.*

1. *Let M^n be an n-manifold and $U \subset M$ be an open subset. Then, U is an n-manifold.*
2. *Let $M^{n_1}, ..., M^{n_k}$ be manifolds of degree $n_1, ..., n_k$, respectively. Then,*

$$M_1^{n_1} \times \cdots \times M_k^{n_k}$$

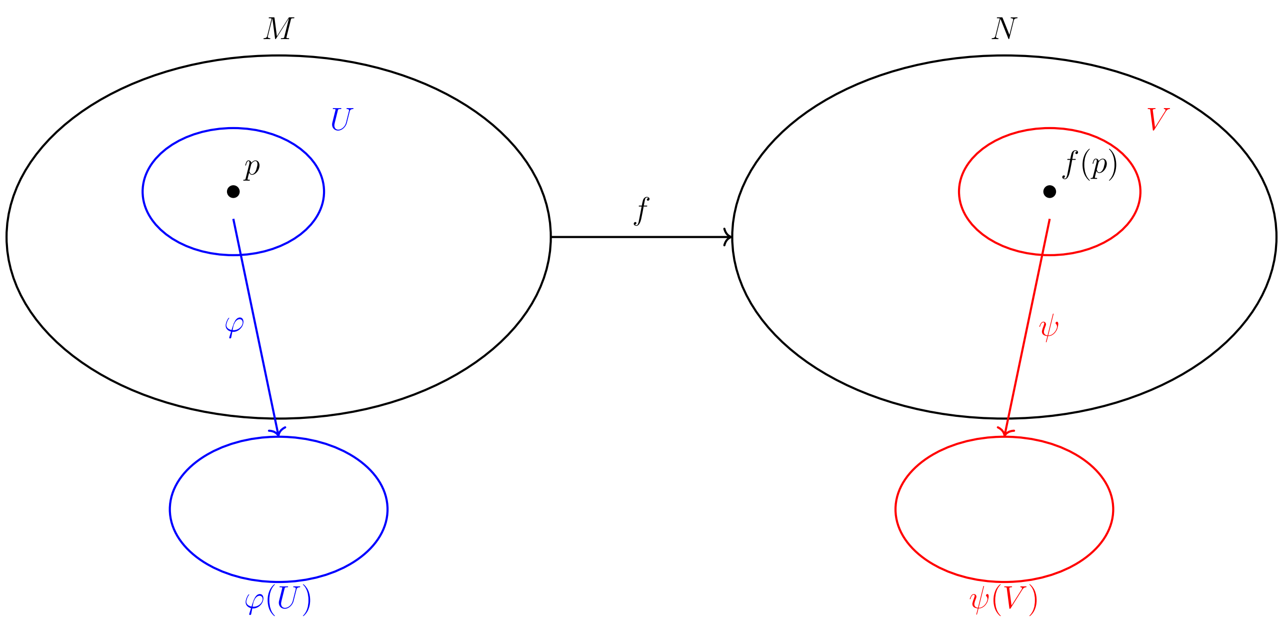
admits a canonical smooth structure of a $n_1 + ... + n_k$ manifold.

Definition 2.6.

2.2 Smooth Functions and Mappings

We now discuss functions between smooth manifolds and some of their properties. We first define what it means for a function on smooth manifolds to be smooth.

Definition 2.7. Let M^m be a smooth manifold, $k \in \mathbb{N}$, and $f : M \to \mathbb{R}^k$ be a function. We say that f is a **smooth function** if for every $p \in M$, there exists a smooth chart (U, φ) on M such that $p \in U$ and the composite $f \circ \varphi^{-1}$ is smooth on the open subset $\tilde{U} \coloneqq \varphi(U) \subset \mathbb{R}^n$.



Remark. The most important special case is that of smooth-real values functions given by

$$f : M \to \mathbb{R}$$

and denote it by $C^\infty(M)$.

Definition 2.8. Let $f : M \to \mathbb{R}^k$ be a function and let (U, φ) be a chart for M. Define the function $\tilde{f} : \varphi(U) \to \mathbb{R}^k$ by $\tilde{f}(x) = f \circ \varphi^{-1}(x)$. We call $\tilde{f}$ the **coordinate representation of f**.

Next, we generalize the idea of a smooth function to maps between smooth manifolds.

Definition 2.9. Let M and N be manifolds of dimension m and n, respectively.

Proposition 2.10. *Every smooth map is continuous.*

Proof. Let M and N be smooth manifolds and suppose that $f : M \to N$ is smooth. Given some $p \in M$, the smoothness of f implies that there are smooth charts (U, φ) containing p and (V, ψ) containing $f(p)$ such that $f(U) \subset V$ and the map $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is smooth. □

Definition 2.11. Let $f : M \to N$ be a function between manifolds M and N. We say that f is smooth if f is smooth at each point $p \in M$. We call f a **diffeomorphism** if the following are true:

1. f is smooth,
2. f is bijective,
3. f^{-1} is smooth.

Proposition 2.12. *Let M, N, P be manifolds. Then the following maps are smooth.*

1. *$c : M \to N$, the constant map is smooth*
2. *$id : M \to N$, the identity is smooth*
3. *If $U \subset M$ is open, then the inclusion*

$$i : U \hookrightarrow M$$

   is smooth.

4. *If $f : M \to N$ and $g : N \to P$ are smooth, then the composition*

$$g \circ f : M \to P$$

   is also smooth.

Remark. We remark that for a diffeomorphism, the smooth and bijective criterion are not enough. Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$. It is clear that f is smooth and bijective. However, its inverse is not smooth at 0. Hence, it does not follow that f is a diffeomorphism from

Proposition 2.13. *We compound some important properties about diffeomorphisms.*

1. *Every composition of diffeomorphisms is a diffeomorphism.*
2. *Every finite product of diffeomorphisms between smooth manifolds diffeomorphism.*
3. *Every diffeomorphism is a homeomorphism and an open map.*

Example 2.14. We now provide some basic examples of smooth maps.

1.

Definition 2.15. Let $g : \mathbb{R}^n \to \mathbb{R}^m$ be smooth and let $x_0 \in \mathbb{R}^n$, then we say that g is **submersion** of x_0 of its differential map

$$D_{x_0}g : \mathbb{R}^n \to \mathbb{R}^n$$

given by

$$D_{x_0}g = \left(\frac{\partial g_i}{\partial x_j}\right)(x_0)$$

is surjective.

Proposition 2.16. *Let $f : M^m \to N^n$ be smooth. Then, f is a*

1. *submersion at $m_0 \in M^m$ if and only if $m \geq n$ and the rank of $D_{\varphi(m_0)}\tilde{f}$ equals n.*

Theorem 2.17. ***(Inverse Function Theorem)*** *Let $U \subseteq \mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ be smooth. Assume that*

$$D_{x_0}f : \mathbb{R}^n \to \mathbb{R}^n$$

for some $x_0 \in \mathbb{R}^n$. Then, there exists a neighborhood $\tilde{U} \subseteq U$ of x_0 such that

$$f|_{\tilde{U}} : \tilde{U} \to \mathbb{R}^n$$

is a diffeomorphism and

$$D_{f(x_0)}(f^{-1}) = \left(D_{f_{x_0}}f\right)^{-1}.$$

Definition 2.18. Let M, N be smooth manifolds and let $f : M \to N$ be smooth. Then, f is a submersion

Definition 2.19. Let $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ be an open cover for a smooth manifold M. A **partition of unity** subordinate to $\mathcal{U}$ is a family of smooth function

$$\{\psi_\alpha : M \to \mathbb{R}\}_{\alpha \in M}$$

such that

1. $\psi_\alpha(x) \subset [0, 1]$ and $\text{supp}(\psi_\alpha) \subset U_\alpha$,
2. the induced family of supposed $\{\text{supp}(\psi_\alpha)\}$ is locally finite and
3. $\displaystyle\sum_{\alpha \in A} \psi_\alpha(p) = 1$ for each $p \in M$.

2.3 Bump Functions

We end this section with a few remarks on bump functions.

Definition 2.20. We call a function Ψ a **bump function** if it is smooth and compactly supported. We create a bump function in the following way. Set

$$f = \begin{cases} e^{-1/t} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Next, set

$$g(t) = \frac{f(t)}{f(t) + f(1-t)}$$

and we can finally set

$$\Psi(t) = g(t+2)g(2-t)$$

Let M be a smooth manifold with (U, φ) as a local chart of M. Let $\psi : U \to \mathbb{R}$ be smooth. Then, $\overline{K} \cdot \varphi$, where $\overline{K} : M \to \mathbb{R}$ is a smooth bump function with $\operatorname{supp}(\overline{K}) \subset U$ is in $C^\infty(M)$ in a neighborhood $p \in V \subseteq U$ and $\overline{K} \cdot \varphi\big|_U = \psi$. Hence, if M is a smooth manifold, U is a chart in M, and $f \in C^\infty(U)$, we can extend to a function $\overline{f} \in C^\infty(M)$ such that $\overline{f}\big|_U = f$.

3 The Derivative of a Function and the Cotangent Space

3.1 The Derivative of a Function

We define the space $C^\infty(M)$ as the set of all smooth functions from M to $\mathbb{R}$. We now motivate the idea of the derivative of a function on a manifold. Let M be a smooth manifold and consider two charts (U, φ) and (V, ψ) such that there exists $p \in M$ such that $p \in U \cap V$. Define $\tilde{f} = f \circ \varphi^{-1}$ and $\overline{f} = f \circ \psi^{-1}$. We have that

$$D_{\varphi(p)}\tilde{f} = \left(\frac{\partial \tilde{f}}{\partial x_1}(\varphi(p)), ..., \frac{\partial \tilde{f}}{\partial x_n}(\varphi(p)) \right)$$

For notational ease, set $\psi \circ \varphi^{-1} := y(x)$ and $\varphi \circ \psi^{-1} = x(y)$. For $i \in \{1, ..., n\}$, we have

$$\frac{\partial \overline{f}}{\partial y_i} = \sum_j \frac{\partial \tilde{f}}{\partial x_j}\bigg|_{\varphi(p)} \frac{\partial x_j}{\partial y_i}\bigg|_{\psi(p)}$$

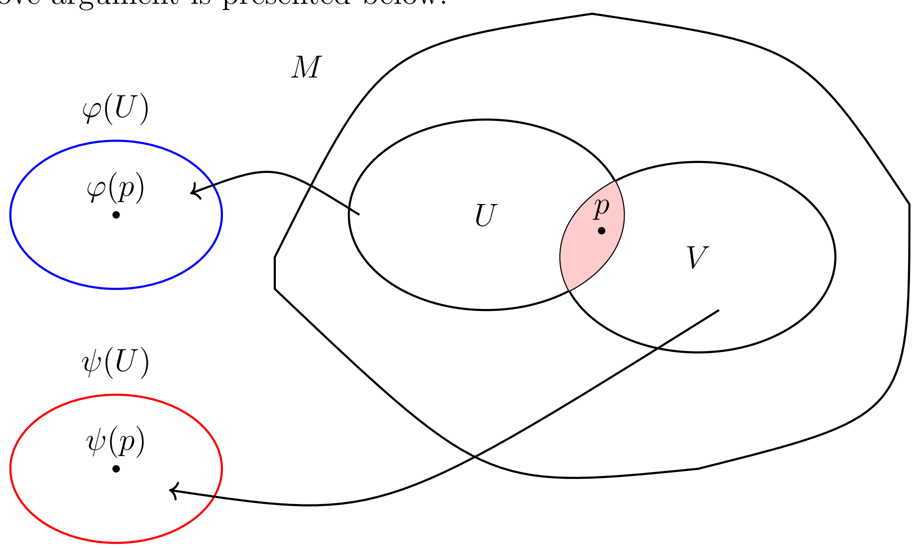
It follows that

$$D_{\psi(p)}\overline{f} = \begin{pmatrix} \dfrac{\partial \overline{f}}{\partial x_1}\bigg|_{\varphi(p)} & \cdots & \dfrac{\partial \overline{f}}{\partial x_n}\bigg|_{\varphi(p)} \end{pmatrix} \cdot \begin{pmatrix} \dfrac{\partial x_1}{\partial y_1}\bigg|_{\psi(p)} & \cdots & \dfrac{\partial x_1}{\partial y_n}\bigg|_{\psi(p)} \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial y_1}\bigg|_{\psi(p)} & \cdots & \dfrac{\partial x_n}{\partial y_n}\bigg|_{\psi(p)} \end{pmatrix}$$

and we arrive at

$$D_{\psi(p)}\overline{f} = D_{\varphi(a)}\tilde{f} \cdot \left(\frac{\partial x_i}{\partial y_j}\right)\Bigg|_{\psi(p)}$$

This calculation demonstrates that $D_p\tilde{f}$ depends on the chart used. An image for the above argument is presented below.



3.2 The Cotangent Space

We now define the cotangent space of a manifold. First, let the subspace $Z_p \subset C^\infty(M)$ be given by

$$Z_p := \{f \in C^\infty \mid D_{\varphi(p)} = 0\}.$$

Note that Z_p does not depend on the choice of charts.

Definition 3.1. Let M be a smooth manifold and $p \in M$. We define the **cotangent space at** p as

$$T_p^*M := C^\infty(M)/Z_p. \tag{3}$$

Proposition 3.2. *Let M be a smooth manifold of dimension n and $p \in M$. Then the following hold:*

1. *The cotangent space T_p^*M is an $n-$dimensional vector space*
2. *If (U,φ) is a coordinate chart with coordinates $(x_1, ..., x_n)$, then the elements $\left(\frac{d}{dx}\Big|_p, ..., \frac{d}{dx_n}\Big|_p\right)$ form a basis for the cotangent space T_p^*M.*
3. *If $f \in C^\infty(M)$ and in the coordinate chart $f \circ \varphi^{-1} = \tilde{f}(x_1, ..., x_n)$, then $D_pf = \sum_I \frac{\partial \tilde{f}}{\partial x_i}(\varphi(p))\frac{d}{dx_i}\Big|_p$ where D_pf is the image of f in the quotient space $C^\infty(f)/Z_p$.*

Proof. Let $\pi_i : \mathbb{R}^n \to \mathbb{R}$ be the i^{th} coordinate projection. Th □

Definition 3.3. Let M be a smooth manifold and $p \in M$. Define the **tangent space at** p given by T_pM as the dual space of T_p^*M. Note that this space does not depend on any charts.

Definition 3.4. A **tangent vector** at $p \in M$ is a linear map $X_p : C^\infty(M) \to \mathbb{R}$ satisfying the Leibniz rule at p:

$$X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$$

Lemma 3.5. *Let M be a smooth manifold and X_p be a tangent vector at p. If $D_pf = 0$, then $X_p(f) = 0$.*

Proof. Let (U, φ) be a chart on M at p. The fundamental theorem of calculus provides

$$\begin{aligned}
\tilde{f}(x) - f(a) &= \int_0^1 \frac{d}{dt}\left(\tilde{f}(\varphi(p)) - t(x - \varphi(p))\right) dt \\
&= \int_0^1 \sum_i \frac{\partial \tilde{f}}{\partial x_i}(\varphi(p)) - t(x - \varphi(p))) \cdot (x_i - \varphi(p))dt \\
&= \sum_i \underbrace{(x_i - \varphi(p))}_{= g_i} \cdot \underbrace{\int_0^1 \frac{\partial \tilde{f}}{\partial x_i}(\varphi(p)) - t(x - \varphi(p)))\, dt}_{= h_i}
\end{aligned}$$

Let $G_i, H_i : M \to \mathbb{R}$ be the globalizations of g_i and h_i, respectfully. Consider a bump function K and define

$$\begin{aligned}
G_i &= \overline{K} \cdot (g_i \circ \varphi) \\
H_i &= \overline{K} \cdot (h_i \circ \varphi)
\end{aligned}$$

It follows that

$$f(x)\bigg|_U = f(p) + \sum_i H_i \cdot G_i$$

□

4 The Tangent Space

We now introduce some basic tools used to study differentiable manifolds. We will first define the tangent space denoted by T_pM, at a point $p \in M$, which we can consider as an analog to the directional derivative of a C^∞ function on a smooth manifold. We will also see that for a smooth function $F : M \to N$, there is an induced linear map between the tangent spaces, $F_* : T_pM \to T_{F(p)}N$, at each point $p \in M$. Associated to a coordinated system around a point p will be a basis of T_pM. Assigning a vector X_p to each point $p \in M$ constructs a vector field on M.

4.1 Tangent Vectors

From elementary calculus, we can imagine what we mean for a tangent vector. How do we extend this idea to a manifold? We first have to content with how we think of elements in $\mathbb{R}^n$. On one hand, we usually think of elements in $\mathbb{R}^n$ as points. On the other, they can also be thought of as vectors.

We first consider a so-called geometric tangent vector in Euclidean space. Given a point $p \in \mathbb{R}^n$, define the **geometric tangent space to $\mathbb{R}^n$ at** p, denoted by $\mathbb{R}^n_p$, as the set $\{p\} \times \mathbb{R}^n := \{(p, v) \mid v \in \mathbb{R}^n\}$. We will write v_p instead of (p, v) as a matter of convenience. Further, define a **geometric tangent vector** in $\mathbb{R}^n$ as an element of $\mathbb{R}^n_p$.

A geometric tangent vector provides us with a means of taking the directional derivative of functions. For any geometric tangent vector $v_p \in \mathbb{R}^n_p$, we have the map

$$D_v|_p : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

given by

$$D_v|_p f = D_v f(a) = \frac{d}{dt} f(p + tv)|_{t=0},$$

which takes the directional derivative in the direction of v at the point p. Note that this operator is linear and satisfies the Leibniz rule.

With the above discussion in mind, we can make the following definition.

Definition 4.1. If p is a point in $\mathbb{R}^n$, a map

$$W : C^\infty(\mathbb{R}^n) \to \mathbb{R}$$

is called a **derivation at** p if W is linear over R and satisfies the Leibniz rule.

Lemma 4.2. *Suppose that $p \in \mathbb{R}^n$, $W \in T_p\mathbb{R}^n$, $f, g \in C^\infty(\mathbb{R}^n)$.*

1. *If f is a constant function, then $Wf = 0$*
2. *If $f(p) = g(p) = 0$, then $W(fg) = 0$.*

Proposition 4.3. *Let $p \in \mathbb{R}^n$.*

1. *For each geometric tangent vector $v_p \in \mathbb{R}^n_p$, the map $D_v|_p : C^\infty \to \mathbb{R}$ defined above is a derivation at p.*
2. *The map $v \mapsto D_v|_p$ is an isomorphism from $\mathbb{R}^n_p$ to $T_p\mathbb{R}^n$.*

Proof. The fact that $D_v|_p$ is a derivation follows from the fact that the directional derivative is linear and abides by the Leibniz rule. To show that the map $v \mapsto D_v|_p$ is an isomorphism, we first note that it is indeed linear.

Injectivity: To show injectivity, suppose that $v_p \in \mathbb{R}^n_p$ has the property that $D_v|_p$ is the zero derivation. We can write v_p as $v_p = v^i e_i|_p$ where e_j is the standard basis vector, and taking f to be the j^{th} coordinate function $x^j : \mathbb{R}^n \to \mathbb{R}$, thought of as a smooth function on $\mathbb{R}^n$, we obtain

$$0 = D_v|_p(x^j) = v^i \frac{\partial}{\partial x^i} x^i|_{x=p} = v^j$$

and hence v_p is the zero function.
Surjectivity: Let $w \in T_p\mathbb{R}^n$ be arbitrary. Define $v = v^i e_i$ such that $v^i = w(x^i)$. We want to show that $w = D_v|_p$. To show this, let f be any smooth real-values function on $\mathbb{R}^n$. We know that we can write f as

$$f(x) = f(p) + \sum$$

□

Corollary 4.4. *For any $p \in \mathbb{R}^n$, the n derivations*

$$\left\{ \frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p \right\}$$

defined by

$$\frac{\partial}{\partial x^i}\bigg|_p f = \frac{\partial f}{\partial x^i}(p)$$

form a basis for $T_p\mathbb{R}^n$, which therefore has dimension n.

4.2 The Tangent Space at a Point

Let M be a smooth manifold of dimension m. We already defined what a C^∞ on an open subset U of M means. This allows us to consider the object $C^\infty(U)$ which is the collection of all C^∞ functions that map

Definition 4.5. For a smooth manifold M and a point $p \in M$, define the **tangent space** to M at a point p, denoted by T_pM, as the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ that satisfies the following two conditions:

1. For all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$, one has

$$X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g) \qquad \text{(linearity)}$$

2. For all $f, g \in C^\infty(p)$, one has

$$X_p(fg) = (X_p f)g(p) + f(p)(X_p(g)) \qquad \text{(Leibniz rule)}$$

with the vector space operations in T_pM defined by

$$(X_p + Y_p)f = X_p f + Y_p f$$
$$(\alpha X_p)f = \alpha(X_p f)$$

A **tangent vector to** M **at** p is any $X_p \in T_pM$.

Lemma 4.6. *Suppose that M is a smooth manifold, $p \in M$, $v \in T_pM$, and $f, g \in C^\infty(M)$. Then we have the following:*

1. *If f is a constant, then $vf = 0$.*
2. *If $f(p) = g(p) = 0$, then $v(fg) = 0$.*

4.3 The Differential of a Smooth Map

If M and N are smooth manifolds and $F : M \to N$ is a smooth map, for each $p \in M$, define the map

$$DF_p : T_pM \to T_{F(p)}N$$

called a **differential of** F **at** p, as follows. Given $v \in T_pM$, let $DF_p(v)$ be the derivation at $F(p)$ that acts on $f \in C^\infty(N)$ by the rule

$$DF_p(v)(f) = v(f \circ F).$$

Note that if $f \in C^\infty(N)$, then $f \circ F \in C^\infty(M)$, so the quantity $v(f \circ F)$ is something that makes sense. We summarize the above discussion in the following theorem.

Theorem 4.7. *Let $F : M \to N$ be a smooth map on manifolds M and N. Then, for $p \in M$, the map $F^* : C^\infty(F(p)) \to C^\infty(p)$ given by $F^*(f) = f \circ F$ is a homeomorphism of algebra and induces a dual vector homomorphism $DF : T_pM \to T_{F(p)}M$ defined by $DF(X_p)f = X_p(F^*f)$*

Remark. The homomorphism $F_* : T_pM \to T_{F(p)}M$ is often called a differential of F. The notation dF, DF, or F' are other common notations for the differential.

Corollary 4.8. *If $F : M \to N$ is a diffeomorphism of M onto an open set $U \subset N$ and $p \in M$, then $DF : T_pM \to T_{F(p)}M$ is an isomorphism onto.*

We know that any open subset of a manifold is a manifold of the same dimension. If (U, φ) is a coordinate chart on M, then the coordinate map induces a isomorphism $D\varphi : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ of the tangent space at each point $p \in U$ onto $T_{\varphi(p)}M$. Additionally, the inverse map $D\varphi^{-1}$ maps $T_{\varphi(p)}\mathbb{R}^n$ isomorphically onto T_pM. The images $e_i = D\varphi^{-1}\frac{\partial}{\partial x^i}$ for $i = 1, .., n$, of the natural basis $\left\{\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^n}\right\}$

4.4 Tangent Bundle

Definition 4.9. Let M be a smooth manifold. We define the **tangent bundle** as the following

$$TM = \bigcup_{p \in M} T_pM. \tag{4}$$

Let M be a smooth manifold with an atlas $\mathcal{U}$. Consider a chart (U, φ). It follows that

$$\begin{array}{ccc} TU = \bigcup_{p\in U} T_pM & \xrightarrow{\hat{\varphi}} & \varphi(U) \times \mathbb{R}^n \\ \downarrow & & \\ = \{p, v_p \mid p \in U, v_p \in T_pM\} & & \end{array}$$

Theorem 4.10. *Let M be a smooth manifold and consider the tangent bundle TM. Then, the following are true.*

1. *TM is a manifold with a canonical smooth structure.*

2. *The map $\pi : TM \to M$ given by $(p, X) \mapsto p$ is a submersion.*

3. *If $p \in M$, then $\pi^{-1}(p) = T_pM$.*

5 Vector Fields

5.1 Basic Definitions

Definition 5.1. Let M smooth manifold. A **vector field** is a smooth mapping $X : M \to TM$ such that

$$\pi \circ X = id_M.$$

We call X a **section** of the tangent bundle given by

$$\Gamma(TM) \coloneqq \{C^\infty(M) \ni X : M \to TM \mid \pi \circ X = id_M\}$$

We have the following diagram between the tangent bundle, the manifold, and $X \in \Gamma(TM)$.

$$\begin{tikzcd} TM \arrow[d, shift left, "\pi"] \\ M \arrow[u, shift left, "X"] \end{tikzcd}$$

Definition 5.2. Let M be a smooth manifold and $X, Y \in \Gamma(TM)$. We define the **Lie bracket** of X and Y as

$$[X, Y] = XY - YX. \tag{5}$$

Proposition 5.3. *Let X, Y, Z be vector fields on a smooth manifold M. Then the following hold:*

1. *$[X, Y]$ is a vector field on M, called the **Lie bracket** of X and Y.*
2. *$[X, Y] = -[Y, X]$*
3. *$[aX + bY, Z] = a[X, Y] + b[Y, Z]$*
4. *$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$*
5. *$[fX, gY] = fg[X, Y] + fX(g) \cdot Y - gY(f) \cdot X$.*

Example 5.4. Let $M = \mathbb{R}$. Consider the vector fields $X = f(t)\dfrac{d}{dt}$ and $Y = g(t)\dfrac{d}{dt}$ for some smooth functions f and g. Let $h \in C^\infty(\mathbb{R})$. Then,

$$X(Y(h)) =$$

5.2 Geometric Understanding of Vector Fields

Definition 5.5. Let M be a smooth manifold. A **one-parameter group diffeomorphism** is a smooth map

$$\gamma : M \times \mathbb{R} \to M$$

such that

1. $\gamma_t : M \to M$ is a diffeomorphism

2. $\gamma_0 = id_M$

3. $\gamma_{s+t} = \gamma_s \circ \gamma_t$ for $s, t \in \mathbb{R}$.

Remark. Let $\text{Diff}(M) \coloneqq \{f : M \to M \mid f \text{ is a diffeomorphism}\}$. Then, the map $(\mathbb{R}, +) \to (\text{Diff}(M), \circ)$ given by $t \mapsto \gamma_t$ is a homeomorphism.

Let γ be a one-parameter group of diffeomorphism and $f \in C^\infty(M)$. Define the map $L : M \to \mathbb{R}$ by

$$L(f)(p) \coloneqq \frac{d}{dt}(f \circ \gamma_t(p)).$$

It follows that L is linear and satisfies the Leibnitz rule. Hence, L is a derivation on M and there is a corresponding vector field X given by

$$X_p(f) = \frac{d}{dt} f(\gamma_t(p))\bigg|_{t=0}.$$

In local coordinates, for a chart (u, φ) with $p \in U$, we have

$$\begin{aligned} X_p(f) &= \frac{d}{dt}(\tilde{f}(\tilde{\gamma}_t(p)))\bigg|_{t=0} \\ &= \frac{\partial \tilde{f}}{\partial x_i}(\varphi(p)) \cdot \frac{d(\tilde{\gamma}_t(p))_i}{dt}\bigg|_{t=0} \\ &= c_i(p) \cdot \frac{\partial}{\partial x_i}\bigg|_{\varphi(p)} \tilde{f} \end{aligned}$$

Definition 5.6. Let X be a vector field on a smooth manifold M. We define an **integral curve** of X as a smooth map

$$\alpha : (a, b) \to M$$

such that $(D_t\alpha)\left(\frac{d}{dt}\right) = X_{\alpha(t)}$.

Example 5.7. Let $M = \mathbb{R}^2$ and consider (x, y) coordinates with the vector field $X = \frac{\partial}{\partial x}$. Then, on an integral curve $\alpha(t) = (x(t), y(t))$, we have that

$$(D_t\alpha)\left(\frac{d}{dt}\right) = \frac{dx}{dt}\frac{\partial}{\partial x} + \frac{dy}{dt}\frac{\partial}{\partial y}.$$

Matching the corresponding components of the vector field X, we have that following system:

$$\begin{cases} \frac{dx}{dt} = 1 \\ \frac{dy}{dt} = 0 \end{cases}$$

which yields the integral curve $\alpha(t) = (t + a, b)$ where (a, b) represent the initial condition of the integral curve.

Theorem 5.8. *Let X be a vector field on a smooth manifold M. For all $p \in M$, there exists a neighborhood $p \in U \subset M$, $\epsilon > 0$, and a unique smooth function $\gamma : U \times (-\epsilon, \epsilon) \to M$ such that*

$$\frac{d\gamma}{dt}(p,t) = X(\gamma(p,t))$$
$$\gamma(p,0) = p.$$

Theorem 5.9. *If M is a smooth compact manifold, then there exists an $\epsilon > 0$ such that the flow is globally defined.*

6 Vector Bundles

6.1 Basic Theory

Definition 6.1. Let M be a topological space. A **real vector bundle of rank k over M** is a topological space E together with a surjective continuous map $\pi : E \to M$ satisfying the following conditions:

1. For each point $p \in M$, the fiber $E_p = \pi^{-1}(p)$ over p is endowed with the structure of a k-dimensional real vector space.
2. For each point $p \in M$, there exists a neighborhood U of p in M and a homeomorphism $\Phi : \pi^{-1}(U) \to U \times \mathbb{R}^k$ (called a **local trivialization of E over U**), satisfying the following conditions:
   - $\pi|_U \circ \Phi = \pi$
   - for each point $q \in U$, the restriction of Φ to E_q is a vector space isomorphism from E_q to $\{q\} \times \mathbb{R}^k \cong \mathbb{R}^k$.

If M and E are smooth manifolds, π is a smooth map, and the local trivializations can be chosen to be diffeomorphisms, them E is called a **smooth vector bundle**. In this case, we call any local trivialization that is a diffeomrophism onto its smooth image a **smooth local trivialization**.

On the intersection $U \cap V$, the map $\psi_U \circ \psi_V^{-1} : (U \cap V) \times \mathbb{R}^n \to (U \cap V) \times \mathbb{R}^n$ is of the form $(\psi_U \circ \psi_V^{-1})(x, v) = (x, g_{UV}(x)(v))$ where g_{UV} is a smooth map from $U \cap V$ to $GL(m, \mathbb{R})$.

6.2 Examples

1. For a smooth manifold M, the tangent bundle TM is a vector bundle.
2. The tangent space of S^1 and $S^1 \times \mathbb{R}$ are isomorphic vector bundles.
3. The Mobius band:

7 The Cotangent Bundle

7.1 Covectors

We begin with an introduction to covectors.

Definition 7.1. Let V be a finite dimensional vector space. Define the **covector on** V to be a real-valued linear functional on V; that is, a linear map $\omega : V \to \mathbb{R}$. The space of all covectors on V is itself is a vector space under the operations of pointwise addition and scalar multiplication. It is denoted by V^* and called the **dual space of V**.

We state an important fact about the dual vector space in the finite dimensional case.

Proposition 7.2. *Let V be a finite dimensional vector space. Given any basis $(E_1, ..., E_n)$ for V, let $\varepsilon^1, ..., \varepsilon^n \in V^*$ be the covectors defined by*

$$\varepsilon^i(E_j) = \delta^i_j,$$

where δ^j_j is the Kronecker delta. Then, $\varepsilon^1, ..., \varepsilon^n$ is a basis for the dual space V^8. Moreover, we have that the dimensions of V and its dual are equal.

Definition 7.3. Suppose that V and W are vector spaces and the map $A : V \to W$ is linear. Define the linear map $A^* : W^* \to V^*$ as the **dual map** by

$$(A^*\omega)(v) = \omega(A(v))$$

for $\omega \in W^*$ and $v \in V$.

Proposition 7.4. *The dual map satisfies the following properties:*

1. $(A \circ B)^* = B^* \circ A^*$.
2. $(Id_V)^* : V^* \to V^*$ *is the identity map of* V^*.

We also have the notion of the **second dual space of** V, denoted by $V^{**} = (V^*)^*$. For each vector space V, there is a natural, basis-independent map $\xi : V \to V^{**}$, defined as follows. For each vector $v \in V$, define a linear functional $\xi(v) : V^* \to \mathbb{R}$ by

$$\xi(v)(\omega) = \omega(v)$$

for $\omega \in V^*$. We have the following property about finite dimensional dual vector spaces.

Proposition 7.5. *For any finite-dimensional vector space V, the map $\xi : V \to V^{**}$ is an isomorphism.*

7.2 Tangent Covectors on Manifolds

Definition 7.6. Let M be a smooth manifold. For each point $p \in M$, define the **cotangent space at** p, denoted by T^*_pM, to be the dual space of T_pM:

$$T^*_p(M) = (T_pM)^*.$$

Elements of T^*_pM are called **tangent coverctors at** p or just **covectors at** p.

8 Tensors

We now generalize the idea of linear maps to multilinear ones– that is, those that take several vectors as inputs and depend linearly on each one separately.

8.1 Multilinear Algebra

In their simplest form, tensors are just real-valued multilinear functions of one or more variables.

Definition 8.1. Suppose that $V_1, ..., V_k$ and W are vector spaces. A map $F : V_1 \times \cdots \times V_k \to W$ is called **multilinear** if it is linear as a function of each variable separately when the other are held fixed: for each i,

$$F(v_1, ..., av_i + a'v_i', ..., v_k) = aF(v_1, ..., v_i, ..., v_k) + a'F(v_1, ..., v_i', ...v_k).$$

Let $L(V_1, ..., V_k; W)$ for the set of all multilinear maps from $V_1 \times \cdots \times V_k$ to W. It is a vector space under the usual operators of pointwise addition and scalar multiplication:

$$(F + F')(v_1, ..., v_k) = F(v_1, ..., v_k) + F'(v_1, ..., v_k)$$

$$(aF)(v_1, ..., v_k) = a(F(v_1, ..., v_k))$$

Example 8.2. (Tensor Products of Covectors). Suppose that V is a vector space and $\omega, \eta \in V^*$. Define the function

$$\omega \otimes \eta : V \times V \to \mathbb{R}$$

given by

$$\omega \otimes \eta(v_1, v_2) = \omega(v_1)\eta(v_2),$$

where the product on the right is ordinary multiplication of real numbers.

The above example can be generalized to arbitrary real-valued multilinear functions as follows: let $V_1, ..., V_k, W_1, ..., W_l$ be real vector spaces, and suppose that $F \in L(V_1, ..., V_k; \mathbb{R})$ and $G \in L(W_1, ..., W_l; \mathbb{R})$. Define a function

$$F \otimes G : V_1 \times \cdots \times V_k \times W_1 \times \cdots \times W_l \to \mathbb{R}$$

given by

$$F \otimes G(v_1, ..., v_k, w_1, ..., w_l) = F(v_1, ..., v_k)G(w_1, ..., w_l).$$

It follows from the multilinearity of F and G that $F \otimes G(v_1, .., v_k, w_1, ..., w_l)$ depends linearly on each argument.

Proposition 8.3. ***(A Basis for the Space of Multilinear Functions)*** *Let $V_1, ..., V_k$ be real vector spaces of dimensions $n_1, ..., n_k$, respectively. For each $j \in \{1, ..., k\}$, let $(e_1^j, ..., e_{n_j}^j)$ be a basis for V_j and let $(\epsilon_j^1, ..., \epsilon_j^{n_j})$ be the corresponding dual basis for V_j^*. Then, the set*

$$\mathbf{B} = \{\epsilon_1^{i_1} \otimes \cdots \otimes \epsilon_k^{i_k} \mid 1 \le i_1 \le n_1, ..., 1 \le i_k \le n_k\}$$

is a basis for $L(V_1, ..., V_k; \mathbb{R})$, which therefore has dimension equal to $n_1 \cdots n_k$.

8.2 Abstract Tensor Products of Vector Spaces

The result of the previous section shows that the vector space of multilinear functions $L(V_1, ..., V_k; \mathbb{R})$ can be viewed as the set of all linear combinations of objects of the form where $\omega^1 \otimes \cdots \otimes \omega^k$ where $\omega^1, ..., \omega^k$ are covectors.

We need to make sense of formal linear combinations. Let S be a set. We can think of formal linear combinations of S as an expression of the form $\sum_{i=1}^{m} a_i x_i$ where a_i are real numbers and x_i are elements of S. We make the following definition:

Definition 8.4. For any set S, a **formal linear combination of elements of** S is a function $f : S \to \mathbb{R}$ such that $f(s) = 0$ for all but finitely many $s \in S$. The **free real variable vector space on** S, denoted by $F(S)$, is the set of all formal linear combinations of elements of S. Under p

Proposition 8.5. ***The Characteristic Property of the Free Vector Space*** *For any set S and any vector space W, every map $A : S \to W$ has a unique extension to a linear map $\tilde{A} : F(S) \to W$.*

Proposition 8.6. ***(The Characteristic Property of the Tensor Product Space)*** *Let U, V, W be finite dimensional vector spaces and let $H : V \times W \to U$ be a bilinear map. Consider the map $\varphi : V \times W \to V \otimes W$ given by*

$$\varphi(v, w) = v \otimes w.$$

Then, there exists a unique function $h =$ such that the following diagram commutes:

$$\begin{array}{ccc} V \otimes W & & \\ \varphi \uparrow & & \\ V \times U & \xrightarrow{H} & U \end{array}$$

Proposition 8.7. ***(Basic Properties of Tensor Products)*** *Let U, V, W be a finite dimensional vector spaces. Then the following hold:*

1. $V \otimes W \cong W \otimes V$
2. $V \otimes (W \otimes U) \cong (V \otimes W) \otimes U$
3. *Let $L : V^* \times W \to Hom(V, W)$ where $(f, w) \mapsto f(v) \cdot w$. Then the induced map $\ell : V^* \otimes W \to Hom(V, W)$ is an isomorphism.*
4. $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.
5. *If $(e_1, ..., e_n)$ is a basis for V and $(e_1, ..., e_m)$ is a basis for W, then $\{e_i \otimes f_j\}$ is a basis for $V \otimes W$.*

Definition 8.8. Let V be a finite-dimensional vector space. We define the **tensor space** $V_{r,s}$ of type (r, s) associated with V to be the vector space

$$V^r \otimes (V^*)^s := \underbrace{V \otimes \cdots \otimes V}_{r \text{ copies}} \otimes \underbrace{V^* \otimes \cdots \otimes v^*}_{s \text{ copies}} \tag{6}$$

for $r, s \geq 0$. We can now define a **tensor** of a vector space V as

$$T(V) := \sum_{r,s\geq 0} V_{r,s} = \left\{ T = \sum_{i=1}^{n} T_{r_i s_i} \;\middle|\; T_{r_j,s_j} \in V_{r_j.s_j} \right\} \tag{7}$$

T_i is a homogenous tensor of degree (r_j, s_j). $T(V)$ is non-commutative, associative, and a graded algebra. We define the operation as $\otimes$ given by

$$u \otimes w := u_1 \otimes \cdots \otimes u_{r_1} \otimes w_1 \otimes \cdots \otimes w_{r_2} \otimes v_1^* \otimes \cdots \otimes v_{s_1}^* \otimes w_1^* \otimes \cdots \otimes w_{s_2}^*.$$

8.3 Symmetric and Alternation Tensors

Definition 8.9. Let V be a finite-dimensional vector space. A covariant k-tensor α on V is said to be **symmetric** if its value is unchanged by interchanging any pair of arguments

$$\alpha(v_1, ..., v_i, ..., v_j, ..., v_k) = \alpha(v_1, ..., v_j, ..., v_i, ..., v_k)$$

whenever $i \leq i \leq j \leq k$.

9 Differential Forms

9.1 Exterior Algebra

Definition 9.1. Let V be a finite dimensional vector space. Define the space $C(V)$ as

$$C(V) := \sum_{k\geq 0} V_{k,0}.$$

In other words, $C(V)$ is the sum of vec

Definition 9.2. Let V and W be finite-dimensional vector spaces. A map $H : V \times \cdots \times V \to H$ is said to be an **alternating map** if

1. H is multilinear
2. $H(v_{\sigma(1)}, ..., v_{\sigma(k)}) = \text{sign}(\sigma) \cdot H(v_1, ..., v_{\sigma(k)})$

Proposition 9.3. ***(Properties of the Wedge Product)*** *Let V_1 and V_2 be finite dimensional vector spaces. Then*

1. $V_1 \wedge V_2 = -V_2 \wedge V_1$

2. *For $\alpha \in \Lambda_r(V)$ and $\beta \in \Lambda_s(V)$, we have $\alpha \wedge \beta = (-1)^{r \cdot s} \beta \wedge \alpha$*

3. *$v_1 \wedge \cdots \wedge v_n = 0$ if $v_i = v_j$ for some $i \neq j$.*

4. *If $(e_1, ..., e_n)$ is a basis for V, then $(e_{i_1} \wedge \cdots e_{i_k})$ with $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ is a basis for $\Lambda_k(V)$.*

5. $\dim(\Lambda_k(V)) = \binom{n}{k}$

6. ***(Universal Property)*** *Let $H : V \times \cdots \times V \to W$ be an alternating map. Then there exists a unique map $h : \Lambda_k(V) \to W$ such that $H = h \circ \varphi$. Moreover, $(\Lambda_k(V), \varphi)$ is unique if there exists a vector space X and a map $\tilde{\varphi}$ such that $(X, \tilde{\varphi})$ satisfies the universal property, then there exists $f : X \to \Lambda_k(V)$ which is an isomorphism such that $\varphi = f \circ \tilde{\varphi}$ and $\tilde{h} = h \circ f$.*

$$\begin{tikzcd}
H : V \times \cdots \times V \arrow[r, "H"] \arrow[d, "\varphi"] & W \\
\Lambda_k(V) \arrow[ur, "h"] \arrow[d, "\tilde{\varphi}"] & \\
X \arrow[uur, "\tilde{h}"] &
\end{tikzcd}$$

9.2 The Algebra of Alternating Tensors

Let V be a finite-dimensional real vector space. Recall that the covariant k-tensor on V is said to be alternating if its value changes sign whenever two arguments are interchanged, or equivalently if any permutation of the arguments causes its value to be multiplied by the sign of the permutation. Alternating covariant k-tensors are also called **exterior forms** or **k-covectors**.

Definition 9.4. The vector space of all k-covectors on V is denoted by $\Lambda^k(V^*)$.

This lemma gives two more characterizations of alternating tensors.

Lemma 9.5. *Let α be a covariant k-tensor on a finite-dimensional vector space V. The following are equivalent:*

1. *α is alternating*

2. *$\alpha(v_1, ..., v_k) = 0$ whenever the k-tuple $(v_1, ..., v_n)$ is linearly dependent*

3. *α gives the value zero whenever two of its arguments are equal.*

For computations with alternating tensors, we adopt the following notation. Given a positive integer k. an ordered $k-$tuple $I = (i_1, ..., i_k)$ of positive integers is called a **multi-index of length** k. If

9.3 The Wedge Product

We want to define a product operation for alternating tensors.

Definition 9.6. (Wedge Product). Let V be a finite-dimensional vector space. Given $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$, we define the **wedge product** to be the following $(k+l)-$covector:

$$\omega \wedge \eta := \frac{(k+l)!}{k!l!} Alt(\omega \otimes \eta). \tag{8}$$

Proposition 9.7. ***(Properties of the Wedge Product)***. *Let V be a finite-dimensional vector space and consider $\omega, \omega', \eta, \eta'$ and ξ be multicovectors on V. Then, we have the following properties:*

1. *Bilinearity: For $a, a' \in \mathbb{R}$,*

$$(a\omega + a'\omega') \wedge \eta = a(\omega \wedge \eta) + a'(\omega' \wedge \eta')$$
$$\eta \wedge (a\omega + a'\omega') = a(\eta \wedge \omega) + a'(\eta' \wedge \omega')$$

2. *Associativity:*

$$\omega \wedge (\eta \wedge \xi) = (\omega \wedge \eta) \wedge \xi$$

3. *Anticommutatitivity: For $\omega \in \Lambda^k(V^*)$ and $\eta \in \Lambda^l(V^*)$,*

$$\omega \wedge \eta = (-1)^{kl} \eta \wedge \omega$$

9.4 Exterior Derivatives

We want to generalize the differential operator on smooth forms, called the **exterior derivative**. To define the exterior derivative on Euclidean space, it is straightforward: given $\omega \in \Omega^k(\mathbb{R}^n)$ such that $\omega = \sum_I \omega_I dx^I$, we define $d\omega$ to be the following $(k+1)$-form

$$d\left(\sum_I \omega_I dx^I\right) = \sum_I d\omega_I \wedge dx^I,$$

where $d\omega_I$ is the differential of the function ω_I. This is flushed out as

$$d\left(\sum_I \omega_I dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = \sum_I \sum_j \frac{\partial \omega_I}{\partial x^j} dx^i \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Example 9.8. Let $f \in \Omega^0(\mathbb{R}) = C^\infty(\mathbb{R})$. The formula above reduces to

$$df = \frac{\partial f}{\partial x^i} dx^i.$$

Example 9.9. Let $\omega \in \Omega^1(\mathbb{R})$ be given by $\omega = \omega_i dx^i$. It follows that

$$\begin{aligned} d(\omega_i dx^i) &= \sum_{i,j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j + \sum_{i>j} \frac{\partial \omega_j}{\partial x^i} dx^i \wedge dx^j \\ &= \sum_{i<j} \left(\frac{\partial \omega_j}{\partial x^i} - \frac{\partial \omega_i}{\partial x^j} \right) dx^i \wedge dx^j \end{aligned}$$

Note that this corresponds to taking the *curl* of the corresponding vector field $X_\omega = \omega_i \frac{\partial}{\partial x^i}$.

Proposition 9.10. ***(Properties of the Exterior Derivative on*** $\mathbb{R}^n$***).***

1. *d is linear over $\mathbb{R}$.*
2. *If $\omega \in \Omega^k(U)$ and $\eta \in \Omega^l(U)$ where $U \subset \mathbb{R}$ is open, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

3. *$d \circ d = 0$.*
4. *d commutes with pullbacks.*

The above properties allow for the application of the exterior derivative to manifolds.

Theorem 9.11. ***(The Exterior Differentiation Theorem)*** *There exists a unique extension $d : \Omega^k(M) \to \Omega^{k+1}(M)$ such that the following properties hold*

1. *d is linear over $\mathbb{R}$.*
2. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, then*

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta.$$

3. *$d \circ d = 0$.*
4. *For $f \in \Omega^0(M) = C^\infty(M)$, df is the differential of f, given by $df(X) = Xf$.*

Proposition 9.12. *Let M and N be smooth manifolds and $F : M \to N$ be a smooth map. Then,*

$$d(F^*(\omega)) = F^*(d\omega) \tag{9}$$

for any $\omega \in \Omega^*(N)$.

Proof. Suppose first that $\omega = f \in \Omega^0(N)$. Then $F^*(f) = f \circ F \in C^\infty(M)$. It follows that $d(F^*(f)) = d(f \circ F)$. So, if $v \in T_pM$, then $d(F^*(f))_p v = d(f \circ F)_p(v) = df(D_pF(x))$. We also calculate $F^*(df)_p(v) = df_{F(p)}(D_pF(v))$. Hence, we have shown the result for a zero form. We move to the general case. Suppose that $\omega \in \Omega^k(N)$. It is enough to argue in the manner above as d and F^* are linear. Since, $d(\cdot)_p$ is determined by the values in a neighborhood of p, we can prove the result in a local chart. Consider the local chart (U, φ) around $F(p)$. Accordingly, we have

$$\omega|_U = \sum_I a_I \cdot dx_{i_1} \wedge \cdots \wedge dx_{i_k}.$$

We can calculate

$$F^*(\omega) = \sum_I F^*(a_I) F^*(dx_{i_1}) \wedge \cdots \wedge F^*(dx_{i_k})$$

and hence

$$\begin{aligned} d(F^*(\omega)) &= \sum_I d(F^*(a_I) \cdot d(F^* x_{i_1}) \wedge \cdots \wedge d(F^* x_{i_k})) \\ &= \sum_I d(F^*(a_I)) \wedge d(F^* x_{i_1}) \wedge \cdots \wedge d(F^* x_{i_k}) \\ &= \sum_I F^*(da_I) \wedge F^*(dx_{i_1}) \wedge \cdots \wedge F^*(dx_{i_k}) \\ &= F^*(d\omega). \end{aligned}$$

□

10 Lie Derivatives

We have already covered how to make sense of directional derivatives of real-valued functions on a manifold. Indeed, a tangent vector $v \in T_pM$ is by definition an operator that acts on a smooth function f to give a number vf that we interpret as a directional derivative of f at p.

What about the directional derivative of a vector field? Let's first consider how this would play out in Euclidean space. It makes sense to define the directional derivative of a smooth vector field X in the direction of a vector $v \in T_p\mathbb{R}^n$. It is the vector

$$D_v X(p) = \frac{d}{dt}\bigg|_{t=0} X_{t+pv} = \lim_{t \to 0} \frac{X_{p+tv} - X_p}{t}.$$

We can easily calculate the directional derivative by applying D_v to each component of W separately:

$$D_v W(p) = D_v W^i(p) \frac{\partial}{\partial x^i}\bigg|_p.$$

This definition is hard to generalize, however. The reason is that we are implicitly using the fact that $\mathbb{R}^n$ is a vector space. That is, the tangent vectors W_{p+tv} and W_p can both be viewed as elements of $\mathbb{R}^n$.

10.1 Lie Derivatives on Vector Fields

Suppose we try to generalize this a manifold M. To begin, we make the replacement of $p + tv$ by a curve $\gamma(t)$ that starts at the point p and whose initial velocity is v. However, this substitution still yields a fundamental error; the vectors $W_{\gamma(t)}$ and $W_{\gamma(0)}$ belong to two different spaces: $T_{\gamma(t)}M$ and $T_{\gamma(0)}M$, respectively. This was negated in the case of $\mathbb{R}^n$ because there is a canonical identification of each tangent space with $\mathbb{R}^n$ itself; but, on a generic manifold there is no such identification. Thus, there is no coordinate independent way

We fix this problem if we replace the vector $v \in T_pM$ with a vector field $X \in$, so we can use the flow of X to push back values of W back to p and then differentiate. We can now make the following definition.

Definition 10.1. Suppose that M is a smooth manifold, X is a smooth vector field on M, and F is the flow of X. For any smooth vector field Y on M, define a rough vector field on M, denoted by the $\mathscr{L}_X Y$ and call the **Lie derivative of Y with respect to X**, by

$$\begin{aligned}(\mathscr{L}_X W)_p &= \frac{d}{dt}\bigg|_p d(F_{-t})_{F_t(p)}(Y_{F_t(p)}) \\ &= \lim_{t\to 0} \frac{d(F_{-t})_{F_t(p)}(Y_{F_t(p)}) - Y_p}{t},\end{aligned}$$

provided the derivative exists. For small $t \neq 0$, at least the difference quotient makes sense: F_t is defined in a neighborhood of p, and F_{-t} is the inverse of F_t, so the objects $d(F_{-t})_{F_t(p)}(Y_{F_t(p)})$ and Y_p are elements of the tangent space T_pM.

Example 10.2. Let $M = \mathbb{R}^2$ and consider the vector fields

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad\qquad Y = \frac{\partial}{\partial x}.$$

Let $p = (x, y)$ and lets calculate the Lie derivative $\mathscr{L}_X Y$. Using the above definition, we need to calculate the flow of X. It follows that we have the system

$$\begin{aligned}\frac{dx}{dt} &= -y \\ \frac{dy}{dt} &= x \\ (x(0), y(0)) &= (x, y).\end{aligned}$$

We can hence write an equation for $x(t)$ as

$$x''(t) + x(t) = 0$$

which has the solution of

$$x(t) = x\cos(t) + y\sin(t).$$

We get that the flow $F_t(x, y)$ is given by

$$F_t(x,y) = \begin{pmatrix}\cos(t) & -\sin(t) \\ \sin(t) & \cos(t)\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix}$$

We have that $Y_{F_t(p)} = \dfrac{\partial}{\partial x}|_{F_t(p)}$

The above definition can be rather tedious. We present an alternate way to calculate Lie derivatives. First, we need the following lemma on the aforementioned Lie brackets.

Lemma 10.3. *Let X, Y be smooth vector fields on a manifold M with or without boundary,and let $X = X^i \frac{\partial}{\partial x^i}$ and $Y = Y^j \frac{\partial}{\partial x^j}$ in terms of the smooth local coordinates (x^i) for M. Then*

$$[X,Y] = (X^i \frac{\partial Y^j}{\partial x^i} - X^j \frac{\partial x^j}{\partial x^i})\frac{\partial}{\partial x^j} = (XY^j - YX^j)\frac{\partial}{\partial x^j}$$

Proof. Because $[X, Y]$ is a smooth vector field, it suffices to check this on a smooth chart. We have

$$\begin{aligned}
[X,Y]f &= X^i \frac{\partial}{\partial x^i}(Y^j \frac{\partial f}{\partial x^j}) - Y^j \frac{\partial}{\partial x^j}(X^i \frac{\partial f}{\partial x^i}) \\
&= X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} + X^i Y^j \frac{\partial^2 f}{\partial x^i \partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i} - Y^j X^i \frac{\partial^2 f}{\partial x^j \partial x^i} \\
&= X^i \frac{\partial Y^j}{\partial x^i}\frac{\partial f}{\partial x^j} - Y^j \frac{\partial X^i}{\partial x^j}\frac{\partial f}{\partial x^i}
\end{aligned}$$

□

Lemma 10.4. *Let V be a smooth vector-field on a smooth manifold M, and let $p \in M$ be a regular point of V. There exists smooth coordinates (s^i) on some neighborhood of p in which V has the coordinate representation $\frac{\partial}{\partial s^1}$.*

Theorem 10.5. *If M is a smooth manifold and X_1, X_2 are vector fields, then $\mathscr{L}_{X_1}X_2 = [X_1, X_2]$.*

Proof. Suppose that V and W are vector fields of a smooth manifold M. Define the set $R(X_1)$ as the set of points $p \in M$ such that $p \in M$ such that $X_1(p) \neq 0$. Note that $R(V)$ is open in M by continuity, and its closure is the support of V. We want to show that

$$(\mathscr{L}_V W)_p = [V, W]_p$$

for all $p \in M$ by considering the following three cases.
Case 1: $p \in R(V)$. In this case, we can choose smooth coordinates (u^i) on a neighborhood of p in which V has the coordinate representation $V = \frac{\partial}{\partial u^1}$ by the lemma. Therefore, the denote the flow of V by $F_t(u) = (u^1 + t, u^2, \dots, u^n)$. Since F_{-t} is just a translation, $d(F_{-t})_{F_t(x)}$ is just the identity at every point $x \in M$. Thus, for any $u \in U$,

$$\begin{aligned}
d(F_{-t})_{F_t(u)}(W_{F_t(u)}) &= d(F_{-t})_{F_t(x)}(W^j(u^1 + t, \dots, u^n)\frac{\partial}{\partial u^j}\bigg|_{F_t(u)}) \\
&= W^j(u^1 + t, \dots, u^n)\frac{\partial}{\partial u^j}\bigg|_u .
\end{aligned}$$

Therefore, by the definition of Lie derivative, we have

$$(\mathscr{L}_V W)_u = \frac{\partial}{\partial t}\bigg|_{t=0} W^j(u^1+t,\ldots,u^n)\frac{\partial}{\partial u^j}\bigg|_u = \frac{\partial W^j}{\partial u^1}(u^1,\ldots,u^n)\frac{\partial}{\partial u^j}\bigg|_u.$$

However, using the lemma we see that

$$[V,W]\bigg|_u = \sum_{j=1}^n (V(W^j)-W(V^j))\frac{\partial}{\partial u^j}\bigg|_u = \sum_{j=1}^n \frac{\partial W^j}{\partial u^1}(u^1,\ldots,u^n)\frac{\partial}{\partial u^j}\bigg|_u$$

Since $V = \frac{\partial}{\partial u^1}$.
Case 2: Let $p \in \operatorname{supp}(V)$, Because the $\operatorname{supp}(V)$ is the closure of $R(V)$, there is a sequence (p_i) that converges to p. By case one, we know that $(\mathscr{L}_V W)_{p_i} = [V,W]_{p_i}$ for every term in the sequence. Thus,

$$(\mathscr{L}_V W)_p = \lim_{i\to\infty}(\mathscr{L}_V W)_{p_i} = \lim_{i\to\infty}[V,W]_{p_i} = [V,W]_p$$

Case 3: $p \in M - \operatorname{supp}(V)$. In this case, $V = 0$ in a neighborhood of p. On one hand, this implies that the flow is equal to the identity map in a neighborhood of p for all t. So,

$$(\mathscr{L}_X W)_p = \frac{d}{dt}\bigg|_p d(F_{-t})_{F_t(p)}(W_{F_t(p)}) = \frac{d}{dt}\bigg|_p W_p = 0$$

since W_p does not depend on t. Also, $[V,W]\big|_p$ is also zero since $V(p) = 0$. $\square$

This theorem provides us with a geometric interpretation of the Lie bracket of two vector fields: it is the directional derivative of the second vector field along the flow of the first. We now present the following properties of the Lie derivative.

Corollary 10.6. *Suppose that M is a smooth manifold and V, W, X are smooth vector fields on M. It follows that*

1. $\mathscr{L}_V W = -\mathscr{L}_W V$
2. $\mathscr{L}_L[V,X] = [\mathscr{L}_V W, X] + [W, \mathscr{L}_V X]$
3. $\mathscr{L}_{[V,W]}X = \mathscr{L}_V\mathscr{L}_W X - \mathscr{L}_W\mathscr{L}_V X$
4. *If $g \in C^\infty(M)$, then* $\mathscr{L}_V(gW) = (Vg)W + g\mathscr{L}_V W$

10.2 Lie Derivatives on Forms

We now consider how to calculate Lie derivatives of differential forms over vector fields. Before we do so, we need to understand how vector fields and differential forms interact with each other.

Definition 10.7. Let M be a smooth manifold and X be a vector field. We define the **interior product** via the map

$$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$$

which sends k forms to $k-1$ forms via the property

$$(\iota_X \omega)(X_1, ..., X_k) = \omega(X, X_1, ..., X_{k-1})$$

for any vector fields $X_1, ..., X_k$ on M.

We briefly mention two important properties of the interior product in the following proposition.

Proposition 10.8. ***(Properties of ι)****: Let M be a smooth manifold. For vector fields X and Y and differential forms $\omega, \alpha, \beta \in \Omega^k(M)$, we have*

1. $\iota_X \iota_Y \omega = -\iota_Y \iota_X \omega$
2. *the Leibniz rule:* $\iota_X(\alpha \wedge \beta) = (\iota_X \alpha) \wedge \beta + (-1)^k \alpha \wedge (\iota_X \beta)$.

We begin with a formal definition as we did for Lie derivatives of vector fields.

Definition 10.9. Let M be a smooth manifold. The **Lie derivative** of a differential form $\omega \in \Omega^k(M)$ respect to a vector field X is defined by

$$\mathscr{L}_X \omega := \left.\frac{d}{dt}\right|_{t=0} F_t^* \omega = \lim_{t \to 0} \frac{F_t^* \omega - \omega}{t}, \tag{10}$$

where F_t is the local flow generated by the vector field X.

Next, we provide a useful identity for Lie derivatives and wedge products of differential forms.

Proposition 10.10. *Suppose that M is a smooth manifold, $X \in$ and $\omega, \eta \in \Omega^*(M)$. Then,*

$$\mathscr{L}_X(\omega \wedge \eta) = (\mathscr{L}_X \omega) \wedge \eta + \omega \wedge (\mathscr{L}_X \eta)$$

We now provide the main result for Lie derivatives on differential forms.

Theorem 10.11. ***(Cartan's Formula)*** *On a manifold M, for any smooth vector field X and any smooth differential form ω,*

$$\mathscr{L}_X \omega = X \tag{11}$$

11 Orientation

We now discuss the topic of orientation. In the case of a line, the orientation is simply which direction in which you traverse it. We need to extend this idea to a manifold, which will be important in the theory of integration on manifolds.

11.1 Orientations of Vector Spaces

We begin with the orientations of vector spaces. The simplest vector spaces are $\mathbb{R}^n$. In the case of $\mathbb{R}^1$, we choose a basis that points to the right (the positive direction). A similar choice is made for $\mathbb{R}^2$. A natural family of preferred ordered bases for $\mathbb{R}^2$ consists of those for which the rotation from the first vector to the second vector is in the counterclockwise direction. Similarly, for $\mathbb{R}^3$, we use the so-called "right-hand rule" to find the direction of the third basis vector.

These methods all have one thing in common: the bases are the ones whose transition matrices from the standard basis have positive determinants. We can generalize to the following definition:

Definition 11.1. Let V be a real-valued vector space of dimension $n \geq 1$. We say that two ordered bases $(e_1, ..., e_n)$ and $(\tilde{e}_1, ..., \tilde{e}_n)$ for V are **consistently oriented** if the transition matrix defined by

$$e_i = B_i^j \tilde{e}_j$$

has a positive determinant.

Definition 11.2. For a vector space V whose dimension is $n \geq 1$, we define an **orientation for V** as an equivalence class of ordered bases.

There is an important connection between orientations and alternating tensors.

Proposition 11.3. *Let V be a vector space of dimension n. Each nonzero element $\omega \in \Lambda^n(V^*)$ determines and orientation $\mathcal{O}_\omega$ of V as follows: if $n \geq 1$, then $\mathcal{O}_\omega$ is the set of ordered bases (e)*

11.2 Orientations of Manifolds

Definition 11.4. Let M be a smooth manifold. We define the **pointwise orientation** of M to be a choice of orientation of each tangent space.

Pointwise orientation is missing some important relations about how the orientations of nearby points relate to each other. We would like there to be a relationship between the orientations and the smooth structure of the manifold.

Definition 11.5. Let M be a smooth manifold of dimension n endowed with some pointwise orientation. If E_i is a local frame of the tangent space of M, we say that E_i is **positively oriented** if $(E_1\big|_p, ..., E_n\big|_p)$ is a positively oriented basis for T_pM. Similarly, we say that we say that E_i is **negatively oriented** if $(E_1\big|_p, ..., E_n\big|_p)$ is a negatively oriented basis for T_pM. A pointwise orientation is said to be **continuous** if every point of M is in the domain of an oriented local frame. An **orientation of** M is a continuous pointwise orientation. An **oriented manifold** is an ordered pair $(M, \mathcal{O})$ where M is an oritentable smooth manifold and $\mathcal{O}$ is a choice of orientation for M.

Example 11.6. We consider a simple example. If M is a zero dimensional manifold, then an orientation on M is a choice of $+1$ or -1 attached to each of its end points.

12 Integration

Our goal is to develop a theory of integration over manifolds. Since we study manifolds by considering charts that live on $\mathbb{R}^n$, we begin with a review of Riemann integration on $\mathbb{R}^n$

12.1 Riemann Integration on $\mathbb{R}^n$

Let $D \subset \mathbb{R}^m$ and $f : A \to \mathbb{R}$ be smooth such that $A \subset \mathbb{R}^n$. For $D \subset A$, define the Riemann integral as

$$\int_D f(x)dx_1 \cdots dx_n \coloneqq \lim_{k_i \to \infty} \sum_{i_1=1}^{k_1} \cdots \sum_{i_n=1}^{k_n} f(x^*_{i_1,\ldots,i_n})\Delta x_1 \cdots \Delta x_n \tag{12}$$

We also have the change of coordinates theorem.

Theorem 12.1. *Let*

12.2 Integration of Differential Forms

On $\mathbb{R}^n$, consider the standard coordinates $(e_1, ..., e_n)$. Then, $de_1 \wedge \cdots \wedge de_n$ gives a vector bundle isomorphism:

$$\Omega^n(\mathbb{R}^n) \to \mathbb{R}^n \times \mathbb{R}.$$

Definition 12.2. Let $D \subset \mathbb{R}$ be the domain of integration and $\omega \in \Omega^n(\mathbb{R}^n)$. Then, we define the integral of ω over the domain D as

$$\int_D \omega \coloneqq \int_D \alpha \; de_1 \wedge \cdots \wedge de_n. \tag{13}$$

Proposition 12.3. *Let $\varphi : V \to A \supset D$ be a diffeomorphism. Then,*

$$\int_{\varphi^{-1}(D)} \varphi^*(\omega) = \pm \int_D \omega$$

for any $\omega \in \Omega^n(A)$.

Proof. Let $\omega \in \Omega^n(A)$. By definition, there exists $\alpha \in C^\infty(M)$ such that $\omega = \alpha \; de_1 \wedge \cdots \wedge de_n$. It follows

$$\begin{aligned} \varphi^*(\omega) &= \varphi^*(\alpha) \cdot \varphi^*(de_1) \wedge \cdots \wedge \varphi^*(de_n) \\ &= (\alpha \circ \varphi) \cdot \det\left(\frac{\partial \varphi_i}{\partial e_j}\right) de_1 \wedge \cdots \wedge de_n \end{aligned}$$

and hence

$$\int_{\varphi^{-1}(D)} \varphi^*(\omega) = \int_{\varphi^{-1}(D)} (\alpha \circ \varphi) \det\left(\frac{\partial \varphi_i}{\partial e_j}\right) de_1 \wedge \cdots \wedge de_n$$

There are now two cases: if $\det\left(\frac{\partial \varphi_i}{\partial e_j}\right) > 0$ or $\det\left(\frac{\partial \varphi_i}{\partial e_j}\right) < 0$. If $\det\left(\frac{\partial \varphi_i}{\partial e_j}\right) > 0$, we can use the change of variables formula. Otherwise, we have that

$$\begin{aligned}\int_{\varphi^{-1}(D)} (\alpha \circ \varphi) \det\left(\frac{\partial \varphi_i}{\partial e_j}\right) de_1 \cdots de_n &= -\int_{\varphi^{-1}(D)} (\alpha \circ \varphi) \left|\det\left(\frac{\partial \varphi_i}{\partial e_j}\right)\right| de_1 \cdots de_n \\ &= -\int_D \omega\end{aligned}$$

where the last equality above follows from the change of coordinates theorem. □

Proposition 12.4. ***(Properties of Integrals of Forms)***. *Suppose that M and N are non-empty oriented smooth n-manifolds and ω and η are compactly supported n- forms on M.*

1. ***Linearity**: If $a, b \in \mathbb{R}$, then*

$$\int_M a\omega + b\eta = a\int_M \omega + b\int_M \eta$$

2. ***Orientation Reversal**: If $-M$ denotes M with the opposite orientation, then*

$$\int_{-M} \omega = -\int_M \omega$$

3. ***Positivity**: If ω is a positively oriented orientation form, then*

$$\int_M \omega > 0$$

12.3 Integration on Manifolds

Definition 12.5. Let M be a smooth oriented $n-$ manifold that is compact. Let $(U_i, \varphi_i)_{i=1}^k$ be an oriented atlas on M and (f_i) be a subordinate partition of unity. Define

$$\int_M \omega := \sum_{i=1}^k \int_{\varphi_i(U_i)} \left(\varphi_i^{-1}\right)^* (f_i \omega)$$

Proposition 12.6. *The definition above for the integral on manifolds is independent of the atlas or the partition of unity.*

Definition 12.7. Let $D \subset M$. We say that D is a **regular domain** if, for each point in M, one of the following holds:

1. there exists an open neighborhood U of p such that $U \subset D$
2. there exists an open neighborhood U of p such that $U \subset M \backslash D$

3. there exists a chart (U, φ) that contains p such that $\varphi(U) \cap \mathbb{H} = \varphi(U \cap D)$, where $\mathbb{H}$ is the upper half plane in n dimensions.

Remark. For a smooth n dimensional manifold M, the boundary ∂M has dimension $n-1$.

Theorem 12.8. ***Stokes' Theorem***: *Let M be a smooth n-dimensional orientable manifold, $D \subset M$ be a regular domain, and ω be an $n-1$-form on M or an open set containing D. Then,*

$$\int_{\partial D} \omega = \int_D d\omega \tag{14}$$

We first discuss the induced orientation on the boundary ∂M. The standard orientation on $\mathbb{R}^n$ is given by the coordinates $(x_1, ..., x_n)$, or equivalently $\left(\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\right)$ is a positively oriented basis for the tangent space $T_p\mathbb{R}^n$. Then, M is oriented if and only if, for each point p, we have that $\left(\left.\frac{\partial}{\partial x_1}\right|_p, ..., \left.\frac{\partial}{\partial x_n}\right|_p\right)$ is an oriented basis for T_pM. We say that $(V_1, ..., V_{n-1})$ is an oriented basis for ∂M if $(n, V_1, ..., V_n)$ is an oriented basis for the tangent space T_pM. We claim that this is well-defined and independent of the choices.

12.3.1 Examples of Stokes' Theorem

We now provide some useful examples of how to apply Stokes' Theorem before we prove the theorem. These formulae will be familiar from a class in vector analysis.

Example 12.9. Let $M = \mathbb{R}$ and $D = [a, b]$ where $b > a$. It follows that $\partial D = \{a, b\}$, a $0-$dimension manifold. We note that the orientation of ∂D comes down to a choice in the sign. Indeed, a basis for the tangent space $T_a\mathbb{R}$ is given by $\partial/\partial x$. We can now apply Stokes' Theorem. It follows that

$$\begin{aligned}\int_{[a,b]} df &= \int_{[a,b]} \frac{\partial f}{\partial x} dx \\ &= \int_{[a,b]} f'(x) dx\end{aligned}$$

and picking the positive orientation of ∂D, we have

$$\int_{\partial D} f = +f(a) - f(b).$$

Combining yields

$$\int_{[a,b]} f'(x)\ dx = f(a) - f(b),$$

the **Fundamental Theorem of Calculus**.

Example 12.10. Let $M = \mathbb{R}^2$ and $\omega \in \Omega^1(\mathbb{R}^2)$. Then, ω has the form $\omega = f(x,y)dx + g(x,y)dy$ for $f, g \in C^\infty(\mathbb{R}^2)$. We calculate the exterior derivative of ω as

$$\begin{aligned}
d\omega &= d(f(x,y)dx + g(x,y)dy) \\
&= \frac{\partial f}{\partial x}dx \wedge dx + \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial g}{\partial x}dx \wedge dx + \frac{\partial g}{\partial y}dy \wedge dx \\
&= \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial g}{\partial x}dx \wedge dy \\
&= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy
\end{aligned}$$

Substituting into Stokes' Theorem yields

$$\int\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \int_{\partial D} f(x,y)dx + g(x,y)dy,$$

which is known as **Green's Theorem**.

Example 12.11. Let $M^2 \subset \mathbb{R}^3$ and $\omega \in \Omega^1(\mathbb{R}^3)$. Then ω has the form $\omega = f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz$ for $f, g, h \in C^\infty(\mathbb{R}^3)$. We calculate the exterior derivative of ω as

$$\begin{aligned}
d\omega &= d\left(f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz\right) \\
&= \frac{\partial f}{\partial y}dy \wedge dx + \frac{\partial f}{\partial z}dz \wedge dx + \frac{\partial g}{\partial x}dx \wedge dz + \frac{\partial g}{\partial z}dz \wedge dy + \frac{\partial h}{\partial x}dx \wedge dz + \frac{\partial h}{\partial y}dy \wedge dz \\
&= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz
\end{aligned}$$

Substituting into Stokes' Theorem yields

$$\begin{aligned}
\int\int_D \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy + \left(\frac{\partial f}{\partial z} - \frac{\partial h}{\partial x}\right) dz \wedge dx + \left(\frac{\partial h}{\partial y} - \frac{\partial g}{\partial z}\right) dy \wedge dz \\
= \int_{\partial D} f(x,y,z)dx + g(x,y,z)dy + h(x,y,z)dz
\end{aligned}$$

which is known as **Stokes' Theorem**.

Example 12.12. Let M and $\omega \in \Omega^2(\mathbb{R}^3)$. Then ω has the form $\omega = f(x,y,z)dx \wedge dy + g(x,y,z)dy \wedge dz + h(x,y,z)dz \wedge dx$ for $f, g, h \in C^\infty(\mathbb{R}^3)$. We calculate the exterior derivative of ω as

$$\begin{aligned}
d\omega &= d\left(f(x,y,z)dx\wedge dy + g(x,y,z)dy\wedge dz + h(x,y,z)dz\wedge dx\right)\\
&= \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dx\right)\wedge dx\wedge dy\\
&\quad + \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy + \frac{\partial g}{\partial z}dx\right)\wedge dy\wedge dz\\
&\quad + \left(\frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial y}dy + \frac{\partial h}{\partial z}dx\right)\wedge dz\wedge dx\\
&= \frac{\partial f}{\partial z}dz\wedge dx\wedge dy + \frac{\partial g}{\partial x}dz\wedge dx\wedge dy + \frac{\partial h}{\partial y}dz\wedge dx\wedge dy\\
&= \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)dx\wedge dy\wedge dz
\end{aligned}$$

Substituting into Stokes' Theorem yields

$$\begin{aligned}
&\int_M \left(\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}\right)dx\wedge dy\wedge dz\\
&= \int_{\partial M} f(x,y,z)dx\wedge dy + g(x,y,z)dy\wedge dz + h(x,y,z)dz\wedge dx
\end{aligned}$$

which is known as the **divergence theorem**.

Remark. In a vector calculus course, these theorems are commonly written using the differential operators *curl* and *divergence*. Although these quantities seem foreign at first, we have shown that they are nothing more than the exterior derivatives of one and two forms in $\mathbb{R}^3$.

12.3.2 Proof of Stokes' Theorem

We now prove Stokes' Theorem in its most general form. Before we do so, we need to prove a more specific result: Stokes' Theorem on p-chains. Let σ be a p-chain and ω be a p-form. We can write $\sigma = \sum_i c_i\sigma_i$, where σ_i is a singular p-simplex. We define the integral of ω over σ as

$$\int_\sigma \omega = \sum_i c_i \int_{\sigma_i} \omega = \sum_i c_i \int_{\Delta^p} \sigma_i^*(\omega). \tag{15}$$

We now state Stokes' Theorem over a p-chain.

Theorem 12.13. *Let M be a compact smooth manifold of dimension n, $\omega \in \Omega^{p-1}(M)$, and σ be a p-chain. Then we have*

$$\int_\sigma d\omega = \int_{\partial\sigma} \omega \tag{16}$$

Proof. Let σ be a p-chain. We can decompose σ into singular p-simplices: $\sigma = \sum_i c_i\sigma_i$. As the integral is linear, it is enough to show the result for an arbitrary singular p-simplex, σ_i.

Case 1: $\mathbf{p=1}$: Let σ be a singular 1-simplex and $\omega \in \Omega^0(M)$. Then, $\omega = f \in C^\infty(M)$. We have that $\sigma : [0,1] \to M$, $\partial\sigma = \sigma(1) - \sigma(0)$, $\Delta^0 = \{0\}$, and $\partial\Delta^1 = \{0\} \cup \{1\}$. It follows that

$$\int_\sigma df = \int_0^1 d(f \circ \sigma) = f \circ \sigma(1) - f \circ \sigma(0)$$

where the last equality follows from the fundamental theorem of calculus in one-dimension. We also calculate that

$$\int_{\partial\sigma} f = f \circ \sigma(1) - f \circ \sigma(0).$$

Hence, we have the result for $p = 1$.
Case 2: $\mathbf{p > 1}$: Let $\omega \in \Omega^{p-1}(M)$. As $\sigma^*(\omega) \in \Omega^{p-1}(\Delta^p)$ has a basis, we can use the standard basis to write

$$\sigma^*(\omega) = \sum_{i=1}^{p-1} f_i \; dx_1 \wedge \cdots \wedge d\hat{x}_j \wedge \cdots \wedge dx_p.$$

It follows that

$$\begin{aligned}
\int_\sigma d\omega &= \int_{\Delta^p} \sigma^*(d\omega) \\
&= \int_{\Delta^p} d(\sigma^*\omega) \\
&= \int_{\Delta^p} \sum_{i=1}^{p-1} \left(\sum_j \frac{\partial f_i}{\partial x_j} dx_j \right) \wedge dx_1 \wedge \cdots \wedge d\hat{x}_j \wedge \cdots \wedge dx_p \\
&= \int_{\Delta^p} \sum_i (-1)^{i-1} \frac{\partial f_i}{\partial x_i} dx_1 \wedge \cdots \wedge dx_p \\
&= \sum_i (-1)^{i-1} \int_{\Delta^p} \frac{\partial f_i}{\partial x_i} \; dx_1 \cdots dx_p
\end{aligned}$$

We now consider $\displaystyle\int_{\partial\sigma} \omega$ and see that □

Proof. Consider $\sigma_\alpha : \Delta^m \to M$ a regular domain with respect to D, oriented $n-$simplices and ∇_α diffeomorphism. Note that ∇_α extends to a C^∞ function in a neighborhood of Δ^n in $\mathbb{R}^n$. Then, Δ^n has one of the following two forms:

1. $\sigma_\alpha(\Delta^n) \subset \text{int}(D)$: Type 1

2.

For any $p \in D$, then there exists a diffeomorphism ∇_α of either type 1 (if $p \in \text{int}(D)$) or type 2 (if $p \in \partial D$) such that $p \in \sigma_\alpha(\Delta^n)$.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be a covering of D such that either $U_\alpha \subset \sigma_\alpha(\Delta^n)$ where σ_α is of type 1 or $U_\beta \cap \sigma_\beta(\Delta^n)$ for σ_β of type 2. The compactness of M implies that there exists a finite subcovering. Accordingly, let $\{f_i\}$ be a subordinate partition of unity. It follows that we can write

$$\int_D d\omega = \sum_i \int_D d(f_i\omega).$$

As $\mathrm{supp}(d(f_i\omega)) \subset \sigma_i(\Delta^n)$, we have

$$\int_D d\omega = \sum_i \int_{\sigma_i} d(f_i\omega) = \sum_i \int_{\partial\sigma_i} d(f_i\omega)$$

where the last equality follows from Stokes' Theorem on simplices. We now have two cases. If σ_i is of type 1, then $f_i\big|_{\partial\sigma_i} = 0$. If σ_i is of type 2, then the only terms on the n^{th} face play a role. So, it follows that

$$\begin{aligned}
\sum_i \int_{\partial\sigma_i} d(f_i\omega) &= \sum_i (-1)^n \int_{\Delta^{n-1}} \left(\sigma_i \circ K_n^{n-1}\right)^* (f_i\omega) \\
&= \sum_i (-1)^n \int_{\Delta^{n-1}} f_i \left(\sigma_i \circ K_n^{n-1}\right) \cdot (\sigma_i \circ K_n^{n-1})^*\omega \\
&= \sum_i (-1)^n (-1)^n \int_{\sigma_i \circ K_n^{n-1}} f_i\omega \\
&= \int_{\partial D} \omega.
\end{aligned}$$

□

13 DeRham Cohomology

Definition 13.1. Let M be a smooth manifold. Consider $\omega \in \Omega^p(M)$. If $d\omega = 0$, we say that ω is **closed**.

Definition 13.2. Let M be a smooth manifold. The exterior derivative is a map $d : \Omega^p(M) \to \Omega^{p+1}(M)$, the image and kernel are linear subspaces. Define the following spaces:

$$\mathcal{Z}^p(M) = \mathrm{Ker}\,(d) = \{\text{closed p-forms on } M\} \qquad \mathcal{B}^p(M) = \mathrm{Im}(d) = \{\text{exact p-forms on } M\}.$$

Definition 13.3. Define the **de Rham cohomology group of degree** to be the quotient vector space

$$H^p_{deR}(M) = \frac{\mathcal{Z}^p(M)}{\mathcal{B}^p(M)} \tag{17}$$

Example 13.4. Let $M = S^1$. Then,

$$H^p_{deR} = \begin{cases} 0 & p \geq 2 \\ \mathbb{R} & p = 1 \\ \mathbb{R} & p = 0. \end{cases}$$

Proof. **Case p = 0**: ☐

Proposition 13.5. *Let M be a compact, connected, oriented $n-$manifold. Then $H^p_{deR}(M) \cong \mathbb{R}$.*

Proof. Let $\alpha \in \Omega^p(M)$. As M is an oriented manifold, there exists $\gamma \in \Omega^n(M)$ such that on a local chart U which is compatible with the orientation, $\gamma|_U = f\ dx_1 \wedge \cdots \wedge dx_n$ such that $f > 0$. It follows that $\displaystyle\int_M \gamma = \sum_{i=1}^m \int_{\varphi(U_i)} \varphi_i f_i dx_1 \wedge \cdots \wedge dx_n > 0$. Define the function $F : \Omega^n(M) \to \mathbb{R}$ be given by

$$\alpha \mapsto \int_M \alpha.$$

From above, we showed that F is a surjective linear map. Let $\alpha \in \Omega^p(M)$ be exact. Then, there exists $\eta \in$ such that $\alpha = d\eta$. It follows that $\displaystyle\int_M \alpha = \int_{\partial M} \eta = 0$. Hence, the set of exact forms lives in the kernel of F. Moreover, F induces a linear surjective map $\tilde{F} : H^n_{deR}(M) \to \mathbb{R}$. ☐

Example 13.6. Let $M = \mathbb{R}^2$ and choose a coordinate system (x, y). Consider the **standard volume form on** $\mathbb{R}^2$ given by $\omega = dx \wedge dy$. We can write this as $d(xdy)$, hence ω is exact. We can switch to polar coordinates by

$$\begin{aligned} x = r\cos(\theta) \quad &\Rightarrow \quad dx = \cos(\theta)dr - r\sin(\theta)d\theta \\ y = r\sin(\theta) \quad &\Rightarrow \quad dy = \sin(\theta)dr + r\cos(\theta)d\theta. \end{aligned}$$

We can then rewrite ω as

$$dx \wedge dy = rdr \wedge d\theta$$

Theorem 13.7. ***(Homology Invariance)*** *Let M and N be a smooth manifolds and consider the map $F : M \times (-\epsilon, 1+\epsilon) \to N$ which is smooth and let $F_t : M \to N$ be such that $F_t(x) = F(x,t)$. Then the induced map on the cohomology*

$$F_t^* : H^p_{deR}(N) \to H^p_{deR}(M)$$

satisfies the property that $F_0^* = F_1^*$.

Corollary 13.8. *$H^p_{deR}(\mathbb{R}^n) = 0$ for $p \geq 1$.*

Proof. Let $F : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ be given by $F(x,t) = t \cdot x$. Let $[\alpha] \in H^p_{deR}(\mathbb{R}^n)$. It follows that $[\alpha] = [F_1^*(\alpha)] = [F_0^*(\alpha)] = 0$. ☐

Corollary 13.9. *For a smooth manifold M, we have $H^p_{deR}(M \times \mathbb{R}^n) \cong H^p_{deR}(M)$ for $p \geq 1$.*

Corollary 13.10.

$$H^p_{deR}(S^n) = \begin{cases} \mathbb{R} & p = 0, n \\ 0 & p \neq 0, n \end{cases}$$

Proof. For the case $p = 0$, S^n is connected. From Proposition **??**, $H^0_{deR}(S^n) = \mathbb{R}$. Similarly, if $p = n$, S^n is connected, compact, and orientable. Again, Proposition **??** yields that $H^n_{deR}(S^n) = \mathbb{R}$.
Consider $S^n = U \cup V$ where U and V are the coresponding stereographic projections: $U = S^n \backslash \{N\}$ and $V = S^n \backslash \{S\}$. We have that $U, V \cong \mathbb{R}^n$. Let α be a closed one-form. Then, $\alpha|_U$ is exact and hence there exists $f \in C^\infty(U)$ such that $\alpha|_U = df$. Similarly, $\alpha|_V$ is exact and hence there exists $g \in C^\infty(V)$ such that $\alpha|_V = dg$. On the intersection $U \cap V$, we have $dg = df$ where $U \cap V = \mathbb{R}^n \backslash \{p\} \cong S^{n-1} \times (0, \infty)$. Then, $d(f - g) = 0$ which provides that $f - g = C$, where C is a constant. Define the map $\tilde{f} : S^n \to \mathbb{R}$ such that $\tilde{f}(x) = f(x)$ if $x \in U$ and $f(x) = g(x) + c$ if $x \in V$. Then we have that $d\tilde{f} = df$ when $x \in U$ and $d\tilde{f} = dg$ when $x \in V$. It follows that $d\tilde{f} = \alpha$.
For $1 < p < n$, we use induction on n. For $n = 1$, we have $H^1_{deR}(S^1) \cong \mathbb{R}$. As above, consider $S^n = U \cup V$ where $U \cap V = \mathbb{R}^n \backslash \{p\} \cong S^{n-1} \times (0, \infty)$ and hence $H^p(U \cap V) \cong H^p(S^{n-1}) = 0$ if $1 \leq p \leq n - 1$. Let $\omega \in \Omega^p(S^n)$ be a closed form, As $U, V \cong \mathbb{R}^n$, there exist $\alpha, \beta \in \Omega^1(U \text{ or } V)$ such that $\omega|_U = d(\alpha)$ and $\omega|_V = d\beta$. Hence, we have that $d\alpha = d\beta$ which yields $d(\alpha - \beta) = 0$. It follows that $\alpha - \beta \in \Omega^p(S^{n-1} \times \mathbb{R})$. As $H^{p-1}_{deR}(S^{n-1} \times \mathbb{R}) = 0$, then $\alpha - \beta = d\eta$ for $\eta \in \Omega^{p-2}(U \cap V)$. Let $U' \subset U$ and $V' \subset V$ be open subsets such that $U' \cap V' \cong S^{n-1} \times (-2, 2)$. Next, let φ be a bump function on $\mathbb{R}$ such that $\operatorname{supp}(\varphi) \subset (-2, 2)$ and $\varphi|_{(-1,1)} = 1$. Let $\tilde{\eta} = \varphi \cdot \eta$ be the extension of φ_η on S^n and $\tilde{U} \subset U'$ and $\tilde{V} \subset V'$ be such that $\tilde{U} \cap \tilde{V} \cong S^{n-1} \times (-1, 1)$. Consider $\tilde{\alpha} = \alpha - d(\varphi_\eta)|_{\tilde{U}}$ □

13.1 Mayor-Viatoris Sequence for the DeRham Cogomology

Definition 13.11. A sequence

$$0 \to A \overset{\alpha}{\to} B \overset{\beta}{\to} C \to 0$$

is **exact** if $\operatorname{Im}(\alpha) = \operatorname{Ker}(\beta)$. At A, α is injective; at B, β is surjective; and at C, $B/\alpha(A) \cong C$. For vector spaces: $B \cong A \oplus C$. This idea is generalized to a **long sequence** when

$$\ldots \to A_k \xrightarrow{f_k} A_{k+1} \xrightarrow{f_{k+1}} A_{k+2} \to \ldots$$

such that $\operatorname{Im}(f_k) = \operatorname{Ker}(f_{k=1}) \subset A_{k+1}$.

Theorem 13.12. *Let M be a smooth manifold and U and V be such that $M = U \cup V$. Then, there exists a long exact sequence*

$$\cdots \to H^{p-1}_{deR} \xrightarrow{i^*, j^*} H^{p-1}_{deR}(U) \oplus H^{p-1}_{deR}(V) \xrightarrow{\pi^*} H^{p-1}_{deR}(U \cap V) \overset{\delta}{\to} H^p_{deR}(M) \to \cdots$$

13.2 The deRham Theorem

Theorem 13.13. ***deRham Theorem*** *Let M be a smooth compact manifold , then the map*

$$H^k_{deR}(M) \to (H_k(M, \mathbb{R}))^*$$

given by

$$[\omega] \mapsto \int_\sigma \omega$$

where σ is a $k-$chain given by $\sigma = a_i \sigma_i$ such that $\sigma_k : \Delta^k \to M$ is an isomorphism.

Consequences of the deRham theorem

1. $H^k_{deR}(M)$ is a finite dimensional vector space.
2. If there exists a basis $\{\sigma_1, ..., \sigma_n\}$ of $H_k(M;\mathbb{R})$, then the cohomology class can be identified by evaluating $\displaystyle\int_{\sigma_1}\omega, ..., \int_{\sigma_n}\omega$ for some ω such that $a = [\omega]$.

13.3 Poincaré Duality

Let M be a compact oriented n-dimensional smooth manifold. Then, there exists a natural isomorphism

$$H^k_{deR}(M) \to H^{n-k}_{deR}(M)$$

associated to the non-singular pairing: $\langle[\omega],[\alpha]\rangle = \int_M \omega\wedge\alpha$. Here, the inner product is given by

$$\langle\cdot,\cdot\rangle : H^k_{deR}(M) \times H^k_{deR}(M) \to \left(H^{n-k}_{deR}(M)\right)^* \cong H^{n-k}_{deR}(M).$$

In local coordinates, consider the $k-$form $\omega = f\ dx^1\wedge\cdots\wedge dx^k$ This construction is coordinate dependent. To have a global construction, we need to introduce a new operator. Consider the following inner product defined by

$$\langle\cdot,\cdot\rangle_p : T_pM \times T_pM \to \mathbb{R}.$$

This naturally induces an oriented orthonormal basis: $\{e_1, ..., e_n\}$. We can consider its dual basis: $\{de_1, ..., de_n\}$ of the cotangent space T^*_pM where $de_I = de_{i_1}\wedge\cdots\wedge de_{i_k}$ of $\Lambda_k(T^*_pM)$. We call the above dual basis the **standard volume form** compatible with orientation and independent of basis.