Vector Spaces

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1 Vector Spaces

Definition 1.1. A **vector space** V over a field $\mathbb{F}$ is a set together with two operations: vector addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \mathbb{F} \times V \to V$ satisfying the following axioms for all $u, v, w \in V$ and $c, d \in \mathbb{F}$:

1. $u + v = v + u$ (commutativity)

2. $(u + v) + w = u + (v + w)$ (associativity)

3. There exists $0 \in V$ such that $v + 0 = v$ (additive identity)

4. For each $v \in V$, there exists $-v \in V$ such that $v + (-v) = 0$ (additive inverse)

5. $c(u + v) = cu + cv$ (distributivity)

6. $(c + d)v = cv + dv$ (distributivity)

7. $c(dv) = (cd)v$ (associativity of scalar multiplication)

8. $1 \cdot v = v$ (multiplicative identity)

Remark. The field that we will always work over is $\mathbb{R} = \mathbb{F}$. In other words, think of scalar multiplication as always being real numbers.

Example 1.2. $\mathbb{R}^n$ with standard addition and scalar multiplication is a vector space over $\mathbb{R}$.

2 Vector Subspaces

Often, vector spaces are too abstract or contain too many vectors to work with. We are interested in finding subsets of the vector space that themselves are vector spaces. We will see that being a vector subspace is rather restrictive. For instance, the zero vector is required for a subset to be a vector subspace.

2.1 Definition of a vector subspace

Definition 2.1. A **subspace** W of a vector space V is a subset $W \subseteq V$ that is itself a vector space under the operations of V. Equivalently, W is a subspace if:

1. $0 \in W$
2. $u, v \in W \implies u + v \in W$
3. $c \in \mathbb{F}, u \in W \implies cu \in W$

Remark. You might ask what the importance of a vector subspace is.

2.2 Examples and non-examples of vector subspaces

We now provide and prove a few examples of vector subspaces.

Example 2.2. The subset

$$W = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 0\}$$

is a vector subspace of the vector space $\mathbb{R}^3$.

Proof. We need to check that the conditions of Definition 2.1 are satisfied.
Condition 1: We have that $0 + 0 + 0 = 0$. Hence, $\vec{0} \in W$.
Condition 2: Let $u, v \in W$. We can write the vectors as

$$u = (x_1, y_1, z_1) \qquad \text{and} \qquad v = (x_2, y_2, z_2).$$

Since $u, v \in W$, we have $x_i + y_i + z_i = 0$ for $i \in \{1, 2\}$. It follows that

$$u + v = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$$

and

$$\begin{aligned}(x_1+x_2)+(y_1+y_2)+(z_1+z_2) &= \underbrace{(x_1+y_1+z_1)}_{=0 \text{ because } u\in W}+\underbrace{(x_1+y_1+z_1)}_{=0 \text{ because } v\in W} \\ &= 0+0 \\ &= 0.\end{aligned}$$

Hence, $u+v\in W$.

Condition 3: Let $u\in W$ and $\alpha\in\mathbb{R}$. We have that

$$\alpha v=(\alpha x,\alpha y,\alpha z)$$

and it follows that

$$\begin{aligned}\alpha x+\alpha y+\alpha z &= \alpha(\underbrace{x+y+z}_{0 \text{ because } u\in W}) \\ &= \alpha\cdot 0 \\ &= 0\end{aligned}$$

Hence, $\alpha u\in W$. Therefore, W is a vector subspace of $\mathbb{R}^3$. □

Example 2.3. The subset

$$W=\{(x,y,z)\in\mathbb{R}^3 \mid ax+by+cz=0\}$$

is a vector subspace of the vector space $\mathbb{R}^3$ for all $a,b,c\in\mathbb{R}$

Proof. Exercise. □

Example 2.4. The subset

$$W=\{(x,y,z)\in\mathbb{R}^3 \mid ax+by+cz=d\}$$

is **not** a vector subspace of the vector space $\mathbb{R}^3$ for all $d\in\mathbb{R}-\{0\}$.

Proof. Note that the zero vector is not in W. □

Example 2.5. The subset

$$W=\{(x,y,z)\in\mathbb{R}^3 \mid xy=0\}$$

is **not** a vector subspace of the vector space $\mathbb{R}^3$.

Proof. The zero vector is clearly in W, since $0\cdot 0=0$. Let $u,v\in W$. It follows that

$$u+v=(x_1+x_2,y_1+y_2,z_1+z_2).$$

We have that

$$\begin{aligned}(x_1+x_2)\cdot(y_1+y_2) &= \underbrace{(x_1y_1)}_{=0}+x_1y_2+x_2y_1+\underbrace{(x_2y_2)}_{=0} \\ &= x_1y_2+x_2y_1 \\ &\neq 0\end{aligned}$$

It follows that $u+v\notin W$ and W is **not** a vector subspace of $\mathbb{R}^3$. □

3 Linear Independence and Bases

3.1 Linear Independence

We ask the following question: given three vectors v_1, v_2, v_3, when is it the case that there exists constants c_1 and c_2 such that $c_1 v_1 + c_2 v_2 = v_3$. This is the idea behind linear independence.

Definition 3.1. A set of vectors $\{v_1, \dots, v_k\} \subset V$ is **linearly independent** if

$$c_1 v_1 + \cdots + c_k v_k = 0 \implies c_1 = \cdots = c_k = 0.$$

If the set is not linearly independent, it is **linearly dependent**.

Example 3.2. Consider the vectors in $\mathbb{R}^3$

$$v_1 = (1,1,0), \quad v_2 = (0,1,1), \quad v_3 = (1,0,1).$$

Form the matrix whose columns are v_1, v_2, v_3:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

To determine whether the vectors are linearly independent, compute the RREF of A. First perform row operations:

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} &\longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \quad (R_2 - R_1) \\
&\longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{bmatrix} \quad (R_3 - R_2) \\
&\longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \quad (\tfrac{1}{2} R_3) \\
&\longrightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{matrix} R_1 - R_3 \\ R_2 + R_3 \end{matrix}
\end{aligned}$$

We arrive at

$$\mathrm{RREF}(A) = I_3.$$

Since there is a pivot in every column, the columns of A are linearly independent. Therefore, the vectors v_1, v_2, v_3 are linearly independent.

We now provide a second way to test for linear independence using the determinant of the corresponding matrix.

Theorem 3.3 (Linear Independence via Determinant). *Let $v_1, v_2, \dots, v_n \in \mathbb{R}^n$, and let*

$$A = [v_1\ v_2\ \cdots\ v_n]$$

be the $n \times n$ matrix whose columns are the vectors $v_1, \dots, v_n$.

Then the vectors $v_1, \dots, v_n$ are linearly independent ***if and only if***

$$\det(A) \neq 0.$$

Remark. Note that this theorem can only be employed for n vectors of length n.

Example 3.4. Let

$$v_1 = (1,0,0), \quad v_2 = (0,1,0), \quad v_3 = (0,0,1) \in \mathbb{R}^3.$$

Form the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since $\det(A) = 1 \neq 0$, the vectors v_1, v_2, v_3 are linearly independent.

Example 3.5. Consider the vectors

$$v_1 = (1,2,3), \quad v_2 = (0,1,4), \quad v_3 = (2,-1,5) \in \mathbb{R}^3.$$

Form the matrix with these vectors as columns:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & -1 \\ 3 & 4 & 5 \end{bmatrix}.$$

Compute the determinant:

$$\det(A) = 1 \cdot (1 \cdot 5 - (-1) \cdot 4) - 0 \cdot (2 \cdot 5 - (-1) \cdot 3) + 2 \cdot (2 \cdot 4 - 1 \cdot 3) = 19 \neq 0.$$

Since $\det(A) \neq 0$, the vectors v_1, v_2, v_3 are linearly independent.

Definition 3.6. Let V be a vector space over a field $\mathbb{F}$ and let $\{v_1, \dots, v_k\} \subset V$.

The **span** of $\{v_1, \dots, v_k\}$ is the set of all linear combinations of these vectors:

$$\operatorname{span}\{v_1, \dots, v_k\} = \{c_1 v_1 + \cdots + c_k v_k \ : \ c_1, \dots, c_k \in \mathbb{F}\}.$$

3.2 The basis

Definition 3.7. A set of vectors $\{v_1, \dots, v_n\}$ is a **basis** of V if

1. $\{v_1, \dots, v_n\}$ is linearly independent
2. $\operatorname{span}\{v_1, \dots, v_n\} = V$

Example 3.8. The most common basis is the **standard basis** for $\mathbb{R}^n$ given by the unit vectors:

$$\begin{aligned}\vec{e}_1 &= (1, 0, ..., 0)\\ \vec{e}_2 &= (0, 1, 0, ..., 0)\\ &\vdots\\ \vec{e}_n &= (0, ..., 0, 1)\end{aligned}$$

Theorem 3.9. *All bases of a vector space V have the same number of elements, called the **dimension** of V.*

3.3 What *really* are bases??

When you first encounter the concept of a **basis**, it can seem abstract, but it is really just a minimal set of vectors that "generate" the whole space. Think of a basis as the building blocks of a vector space: every vector in the space can be written as a combination of these building blocks, and no block is redundant.

3.3.1 How to think about bases

1. **Spanning the Space:** A basis must span the vector space. This means that by taking linear combinations of the basis vectors, you can reach any vector in the space. For example, in $\mathbb{R}^2$, the vectors

$$e_1 = (1, 0), \quad e_2 = (0, 1)$$

span $\mathbb{R}^2$, because any vector (x, y) can be written as

$$(x, y) = xe_1 + ye_2.$$

2. **No Redundancy (Linear Independence):** The vectors in a basis must be linearly independent. This ensures that no vector in the basis can be written as a combination of the others. In $\mathbb{R}^2$, the vectors

$$v_1 = (1, 0), \quad v_2 = (2, 0)$$

do *not* form a basis, because v_2 is just $2v_1$, so there is redundancy.
3. **Minimal Set:** A basis is the smallest set of vectors needed to span the space. If you add more vectors than necessary, some will be dependent and no longer part of a proper basis.

3.3.2 Examples

- **Standard Basis of $\mathbb{R}^3$:**

$$e_1 = (1, 0, 0), \quad e_2 = (0, 1, 0), \quad e_3 = (0, 0, 1)$$

  is a basis because it spans all of $\mathbb{R}^3$ and the vectors are independent.

- **Non-Standard Basis of $\mathbb{R}^2$:**

$$v_1 = (1, 1), \quad v_2 = (1, -1)$$

is also a basis. Any vector (x, y) can be written as

$$(x, y) = \frac{x+y}{2} v_1 + \frac{x-y}{2} v_2.$$

This shows that bases are not unique: there are many different sets of vectors that can serve as a basis for the same space.

- **Basis for a Subspace:** Consider the subspace of $\mathbb{R}^3$ defined by

$$W = \{(x, y, z) : x + y + z = 0\}.$$

One basis for W is

$$v_1 = (1, -1, 0), \quad v_2 = (1, 0, -1),$$

because these two vectors are independent and any vector in W can be written as a combination of v_1 and v_2. Notice that the dimension of this subspace is 2, even though it sits inside $\mathbb{R}^3$.

3.3.3 Intuition

A good way to think about bases is as *directions you can move in.* In $\mathbb{R}^2$, two independent directions suffice; in $\mathbb{R}^3$, three are needed. For subspaces, the basis tells you the minimum number of independent directions that fully describe that subspace. Once we have a good grasp, other topics in linear algebra—like coordinates, dimension, and rank—become much more intuitive.

3.4 Why bases matter

Understanding bases is not just a theoretical exercise—it is extremely useful in practice. A basis allows us to **describe every vector in a space in a systematic, compact way**. Instead of working with vectors in their raw form, we can work with their *coordinates relative to a basis.* This abstraction makes computations, transformations, and understanding of the space much easier.

Example 3.10. Suppose we are working in $\mathbb{R}^2$, but instead of the standard basis

$$e_1 = (1, 0), \quad e_2 = (0, 1),$$

you choose the basis

$$v_1 = (1, 1), \quad v_2 = (1, -1).$$

Any vector $(x, y) \in \mathbb{R}^2$ can now be written as a linear combination of v_1 and v_2:

$$(x, y) = c_1 v_1 + c_2 v_2,$$

where

$$c_1 = \frac{x+y}{2}, \quad c_2 = \frac{x-y}{2}.$$

Why this is useful: If a linear transformation has a simple form in this basis, we can compute it more easily. For example, suppose a transformation reflects across the line $y = x$. In the v_1, v_2 basis, this reflection just swaps the coefficients c_1 and c_2, instead of needing a more complicated formula in the standard basis. Bases allow us to *compress information*: instead of storing vectors as raw numbers, we can store them as coordinates relative to a basis. This is how graphics engines, data compression, and many algorithms in engineering and science work.

Example 3.11. Consider a subspace $W \subset \mathbb{R}^3$ defined by

$$W = \{(x, y, z) : x + y + z = 0\}.$$

Here is the procedure to find a basis for the subspace W. Since W represents a plane which is of degree 2. Hence, our basis should have *two* vectors. First, isolate x as $x = -y - z$ which allows us to the write the plane as

$$(x, y, z) = (-y - z, y, z).$$

The key to finding the two basis vectors lies in factoring the above expression as

$$(-y - z, y, z) = y(-1, 1, 0) + z(-1, 0, 1)$$

We then have a basis for W as

$$v_1 = (1, -1, 0), \quad v_2 = (1, 0, -1).$$

Now, any vector in W can be written as $c_1 v_1 + c_2 v_2$. If you want to solve $Ax = b$ where $x \in W$, you only need to solve for c_1 and c_2, reducing the problem from three unknowns to two. This makes computations simpler and gives insight into the structure of the solution space.

Takeaway

Working with vectors in terms of a basis is like choosing a good coordinate system: it lets you *simplify problems, see patterns, and perform computations efficiently.* In higher-dimensional spaces, where vectors can have dozens, hundreds, or even thousands of components, working in the right basis is essential for both understanding and computation.

4 Column Space, Row Space, and Null Space

When studying a matrix A, two important subspaces arise naturally: the **column space** and the **null space**. The column space, $\text{Col}(A)$, consists of all linear combinations of the columns of A, and it captures all vectors that can be produced by the matrix acting on a vector. The null space, $\text{Null}(A)$, consists of all vectors x satisfying $Ax = 0$, representing the directions in which the matrix "flattens" the space to zero. Let A be an $m \times n$ matrix over $\mathbb{R}$. We have the following three definitions.

4.1 The basic subspaces

Definition 4.1. The **column space** of A, denoted $\mathrm{Col}(A)$, is the span of the columns of A:

$$\mathrm{Col}(A) = \mathrm{span}\{\text{columns of } A\} \subseteq \mathbb{R}^m$$

The **row space** of A, denoted $\mathrm{Row}(A)$, is the span of the rows of A:

$$\mathrm{Row}(A) = \mathrm{span}\{\text{rows of } A\} \subseteq \mathbb{R}^n$$

The **null space** of A, denoted $\mathrm{Null}(A)$, is the set of all solutions to $Ax = 0$:

$$\mathrm{Null}(A) = \{x \in \mathbb{F}^n : Ax = 0\} \subseteq \mathbb{R}^n$$

4.2 Rank and Nullity

To understand these spaces efficiently, we focus on their **bases**, which give a minimal set of vectors that generate the entire space. The number of vectors in a basis of the column space is called the **rank** of the matrix, and it measures how many independent directions the matrix can produce. Similarly, the number of vectors in a basis of the null space is called the **nullity**, and it measures the number of independent directions that get sent to zero.

Definition 4.2. The **rank** of A, $\mathrm{rank}(A)$, is the dimension of the column space of A.

Definition 4.3. The **nullity** of A, $\mathrm{nullity}(A)$, is the dimension of the null space of A.

The concepts of rank and nullity are intimately connected through the **Rank-Nullity Theorem**, which provides a fundamental relationship between the dimension of the domain of a matrix, the rank, and the nullity.

Theorem 4.4 (Rank-Nullity Theorem)**.** *For an $m \times n$ matrix A,*

$$\mathrm{rank}(A) + \mathrm{nullity}(A) = n$$

4.3 What do we do with these things??

In linear algebra, when studying spaces such as the column space, row space, or null space of a matrix, we rarely need to consider *all* vectors in the space. Instead, we focus on a **basis**, because a basis captures all the essential information in the smallest possible set of vectors. Every vector in the space can be expressed as a linear combination of the basis vectors, so knowing the basis is equivalent to knowing the entire space. This is extremely useful for computations: for example, when solving systems of equations, understanding the rank of a matrix, or describing solutions to $Ax = b$, we only need the basis vectors and their coordinates. By working with a basis rather than the full space, we simplify problems, reduce redundancy, and gain a clearer understanding of the structure of the space, without losing any information.

Example 4.5 (Column Space)**.** Let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 5 & 11 \end{bmatrix}.$$

Step 1: Calculate REF:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 2 & 5 & 11 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 2: Identify pivot columns. Here, columns 1, 2, and 3 are pivot columns.
Step 3: Choose the corresponding columns from the original matrix as a basis:

$$\mathrm{Col}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 4 \\ 11 \end{bmatrix} \right\}.$$

Example 4.6 (Row Space)**.** For the same matrix A, the row space basis comes from the nonzero rows of the *row-reduced echelon form* (RREF):

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & 0 & -5 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{bmatrix}.$$

Step 1: Take the nonzero rows as a basis:

$$\mathrm{Row}(A) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 0 & -5 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 4 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \right\}.$$

Step 2 (Optional): You can also take the rows of the original matrix that correspond to pivot positions in RREF as an alternative basis.

Example 4.7 (Column Space with Dependent Columns)**.** Consider

$$B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 0 & 1 & 1 \end{bmatrix}.$$

Step 1: Calculate RREF:

$$\mathrm{RREF}(B) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Step 2: Identify pivot columns. Here the pivot columns are 1 and 3.
Step 3: Basis for column space:

$$\mathrm{Col}(B) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} \right\}.$$

Step 4: Basis for row space: Take nonzero rows of RREF:

$$\mathrm{Row}(B) = \mathrm{span}\left\{ \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 1 \end{bmatrix} \right\}.$$

Example 4.8. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 2 & 4 & 1 & 3 & 7 \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}.$$

We want to calculate the rank, nullity, and bases for the null space, row space, and column space.

Part 1: Compute the RREF. We calculate the RREF of A as:

$$\text{RREF}(A) = \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Part 2: Determine the Rank and Nullity. From step 1, we see that the pivot columns are columns 1 and 3. We know that the rank of A is the number of pivot columns. Hence, $\text{rank}(A) = 2$. To find the nullity, we can employ two tools: the rank nullity theorem or counting the number of free variables. Using the former approach, we see the that the number of columns is 5, so the nullity is

$$\text{nullity}(A) = 5 - 2 = 3.$$

Part 3: Basis for the Column Space. The basis for the column space is given by the pivot columns of the original matrix. Hence, we have

$$\text{Col}(A) = \text{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Part 4: Basis for the Row Space. The basis for the row space is given by the nonzero rows of the RREF:

$$\text{Row}(A) = \text{span}\left\{ \begin{bmatrix} 1 & 2 & 0 & 1 & 3 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \end{bmatrix} \right\}.$$

Part 5: Basis for the Null Space. Let $x = (x_1, x_2, x_3, x_4, x_5)^T$ be in the null space. From the RREF, we have

$$x_1 + 2x_2 + x_4 + 3x_5 = 0, \quad x_3 + x_4 + x_5 = 0.$$

We next solve for the pivot variables x_1 and x_3 in terms of the free variables x_2, x_4, x_5:

$$x_1 = -2x_2 - x_4 - 3x_5, \quad x_3 = -x_4 - x_5.$$

To ease notation, we define the free variables as $x_2 = s$, $x_4 = t$, $x_5 = u$. Then

$$x = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + u \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

Thus, a basis for the null space is

$$\mathrm{Null}(A) = \mathrm{span}\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$