The Laplace Transform

Contents

1 Introduction to the Laplace transform

We have reached one of the most interesting and useful parts of studying differential equations: the Laplace transform. This is a method for solving the types of equations of which we have concerned ourselves this term. The Laplace transform is crucial in many areas of engineering including signals, mechanical vibrations, and circuits. Accordingly, it is the preferred method for most engineers.

1.1 Definition of the Laplace transform

Definition 1.1. The **Laplace transform** of f is the function $F(s)$ defined by

$$\boxed{\mathscr{L}\{f(t)\}(s) = F(s) := \int_0^\infty f(t)\, e^{-st}\, dt} \tag{1}$$

for all complex numbers $s \in \mathbb{C}$ for which the integral converges. The set of such s is called the **region of convergence** of the transform.

Definition 1.2. A function f is of **exponential type** if exist constants $M, a > 0$ such that $|f(t)| \leq Me^{at}$ for sufficiently large t.

These are the main types of functions that we will concern ourselves with for this course. In words, the Laplace transform assigns to each function $f(t)$ a new function $F(s)$ obtained by integrating $f(t)$ against the decaying exponential e^{-st}.

1.2 Historical background

The Laplace transform is named after the French mathematician **Pierre-Simon Laplace** (1749–1827), who introduced related integral techniques in his work on probability theory and celestial mechanics. In particular, Laplace used transforms of the form

$$\int_0^\infty e^{-st} f(t)\, dt$$

to study generating functions and to simplify complex differential equations arising in physics.

However, the essential ideas behind the Laplace transform predate Laplace himself. **Leonhard Euler** and **Joseph-Louis Lagrange** had already explored exponential kernels and integral representations in the 18th century. Laplace's contribution was to systematically apply these ideas and recognize their unifying power.

In the 19th century, the transform was further developed in the context of differential equations and operational calculus, particularly by **Oliver Heaviside**. Heaviside used transform-like methods in engineering problems (especially electrical circuits), often in a formal and non-rigorous way. His work was later placed on a rigorous foundation through developments in complex analysis and functional analysis in the early 20th century.

1.3 Motivation and connection to power series

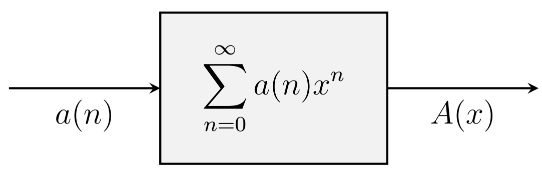
To most people who study it, the Laplace transform is this rather obscure object and its origins are always somewhat unclear. We will try to dispel this mystery by showing a natural derivation from power series. Recall that a power series has the form

$$\sum_{n=0}^{\infty} a_n x^n = A(x).$$

Here, let's make a slight modification to the form of a power series. We recall from calculus two that a series a_n is a discrete function that accepts all positive integers and returns some real number. Ergo, its natural to write a power series as

$$\sum_{n=0}^{\infty} a(n) x^n = A(x).$$

From this perspective, it's natural to think of power series as accepting a function in terms of the discrete variable n and returning a function in terms of the continuous variable x. This is shown pictorially below:



To get a better feel for this, let's consider a few examples:

$$\begin{array}{rcl} a(n) & \rightsquigarrow & A(x) \\ 1 & \rightsquigarrow & \dfrac{1}{1-x} \quad \text{for} \quad |x| < 1 \\ \dfrac{1}{n!} & \rightsquigarrow & e^x \quad \text{for all } x \end{array}$$

So, how do power series lead us to the Laplace transform? The answer comes from replacing the discrete variable n by a continuous variable t. In other words, we are going to consider the *continuous analog* of a power series. The main– and only difference aside from some cosmetic changes to come– is that we are going to replace the discrete sum by its continuous analog: the integral.

Hence, we replace the sum by the integral and copy the rest of the summand:

$$\sum_{n=0}^{\infty} a(n) x^n \quad \rightsquigarrow \quad \int_0^{\infty} a(t) x^t \, dt.$$

Turning the above integral to the Laplace transform (1) now *solely* relies on a series of cosmetic changes. First, the above integral will involve integrating functions involving the exponential x^t. Here, the integral will spit out a function in terms of x. There are some natural restrictions that we should impose on this variable. First, if $x > 1$, this integral is not very likely to converge. Remember: we are working with improper integrals here; they require delicate handling. We also want $x > 0$. If x were -1, for example, the integral would cease to be real. Hence, we restrict the variable x to the following domain:

$$0 < x < 1$$

No engineers– and very few mathematicians– would like to integrate an integrand of the form x^t. The only object people like to integrate is the exponential whose base is e. Accordingly, we write

$$x^t = e^{\log(x)\cdot t}.$$

Since the variable x is now $\log(x)$, we have

$$0 < s < 1 \Rightarrow -\infty < \log(x) < 0$$

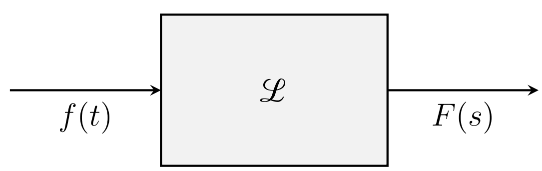
We now have a variable of $\log(x)$. Since no one works with a variable called $\log(x)$, let's set $s = -\log(x)$. The negative is to change the domain of s to $(0, \infty)$. Finally, we called the input function $a(t)$. Most people call this function $f(t)$, so we will change it to this, too. Making these cosmetic changes, we have

$$\int_0^\infty f(t)e^{-st}\,dt = F(s),$$

which matches the definition for the Laplace transform in (1).

1.4 The Laplace transform as an integral transform

The Laplace transform is what is known as an **integral transform**. Think of it in terms of the following schematic:



The above schematic provides an important fact: the Laplace transform accepts a function of t and outputs a function of s. This is why the Laplace transform is called a *transform* and not an operator; transforms change the independent variable. This is something that takes some getting used to. The analogy to the power series is useful for understanding this fact. In fact, power series are sometimes called a z-transform for this exact reason.

Proposition 1.3. *The Laplace transform is linear. In particular, it satisfies the following identities:*

1. $\mathscr{L}(f+g) = \mathscr{L}(f) + \mathscr{L}(g)$ *for all functions f and g of exponential type*
2. $\mathscr{L}(cf) = c\mathscr{L}(f)$ *for all functions f of exponential type and $c \in \mathbb{R}$.*

Proof. The above identities easily following from the linearity of the indefinite integral. I leave it for the reader to verify the above two identities. □

2 Calculating the Laplace transform

We now calculate the main Laplace transforms that we will need to solve differential equations in the forthcoming sections. We start with calculating the Laplace transforms of various common functions we have encountered thus far in the course.

2.1 Laplace transforms of basic functions

We begin with calculating the Laplace transform of the constant function $f(t) = 1$. Substituting $f(t)$ into the definition in (1) yields the following:

$$\begin{aligned} \mathscr{L}(1) &= \int_0^\infty 1 \cdot e^{-st}\, dt \\ &= \lim_{R\to\infty} \int_0^R e^{-st}\, dt \\ &= \lim_{R\to\infty} \frac{e^{-sR} - 1}{-s} \\ &= \frac{1}{s}. \end{aligned}$$

The above calculation, however, is wrong. The first three equalities are valid. The last equality is not always true; it is *only* true for $s > 0$. Therefore, we have our first Laplace transform:

$$\boxed{\mathscr{L}(1) = \frac{1}{s}, \quad s > 0} \tag{2}$$

The next obvious function whose Laplace transform is of concern is the exponential function $f(t) = e^{at}$. Instead, let's find the Laplace transform of a more general function $e^{at}f(t)$. This is called the **exponential shift law**.

Proposition 2.1 (The Exponential Shift Law)**.** *Suppose that $f(t)$ has a Laplace transform $F(s)$ for $s > 0$. Then,*

$$\boxed{\mathscr{L}\left(e^{at}f(t)\right) = F(s-a), \quad s > a} \tag{3}$$

Proof. From the definition of the Laplace transform, we calculate

$$\begin{aligned} \mathscr{L}\left(e^{at}f(t)\right) &= \int_0^\infty e^{at}f(t)e^{-st}\, dt \\ &= \int_0^\infty f(t)e^{-(s-a)}\, dt. \end{aligned}$$

The last integral is simply the Laplace transform with the variable $s - a$ instead of s. This yields the result. □

The above proposition immediately gives us the Laplace transform for the exponential function $f(t) = e^{at}$.

$$\boxed{\mathscr{L}\left(e^{at}\right) = \frac{1}{s-a}, \quad s > a} \tag{4}$$

This formula works for complex values of a, too. This fact allows us to calculate the Laplace transforms of sinusoidal functions. From Euler's formula, we have

$$\begin{aligned}\sin(at) &= \frac{e^{it} - e^{-it}}{2i}\\ \cos(at) &= \frac{e^{it} + e^{-it}}{2}\end{aligned}$$

Let's calculate the Laplace transform of $\sin(at)$. We have

$$\begin{aligned}\mathscr{L}(\sin(at)) &= \mathscr{L}\left(\frac{e^{iat} - e^{-iat}}{2i}\right)\\ &= \frac{1}{2i}\left\{\mathscr{L}\left(e^{iat}\right) - \mathscr{L}\left(e^{-iat}\right)\right\}\\ &= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right)\\ &= \frac{a}{s^2+a^2}.\end{aligned}$$

Therefore, we have

$$\boxed{\mathscr{L}(\sin(at)) = \frac{a}{s^2+a^2}, \quad s > a} \tag{5}$$

We have a similar calculation for the Laplace transform of $\cos(at)$:

$$\begin{aligned}\mathscr{L}(\cos(at)) &= \mathscr{L}\left(\frac{e^{iat} + e^{-iat}}{2}\right)\\ &= \frac{1}{2}\left\{\mathscr{L}\left(e^{iat}\right) + \mathscr{L}\left(e^{-iat}\right)\right\}\\ &= \frac{1}{2}\left(\frac{1}{s-ia} + \frac{1}{s+ia}\right)\\ &= \frac{s}{s^2+a^2},\end{aligned}$$

and hence

$$\boxed{\mathscr{L}(\cos(at)) = \frac{s}{s^2+a^2}, \quad s > a.} \tag{6}$$

Finally, we calculate the Laplace transform of a polynomial. Consider the function $f(t) = t^n$. Calculating yields

$$\begin{aligned}\mathscr{L}(t^n) &= \int_0^\infty t^n e^{-st}\,dt\\ &= \lim_{R\to\infty}\left[\frac{t^n e^{-st}}{-s}\right]\Bigg|_0^R + \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}\,dt\\ &= \frac{n}{s}\mathscr{L}(t^{n-1}),\end{aligned}$$

which gives a recursive formula for the Laplace transform of t^n. Iterating n times and using that the Laplace transform of 1 is $\frac{1}{s}$, we have

$$\boxed{\mathscr{L}(t^n) = \frac{n!}{s^{n+1}}, \quad s > 0} \tag{7}$$

2.2 Laplace transforms of differential equations

Now that we have understood how to calculate the Laplace transforms of various common functions, we need to understand how to apply it to differential equations. To that end, we need to understand how Laplace transforms interact with derivatives. Consider the Laplace transform of the first derivative $f'(t)$. Applying definition (1),

$$\mathscr{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\, dt.$$

Whenever the integrand of an integral contains a derivative, the natural process to apply is integration by parts. Indeed, if we select

$$u = e^{-st}, \quad dv = f'(t)\, dt,$$

so that

$$du = -se^{-st}\, dt, \quad v = f(t).$$

Applying the integration by parts formula yields that

$$\mathcal{L}\{f'(t)\} = \left[e^{-st} f(t)\right]_0^\infty + s\int_0^\infty e^{-st} f(t)\, dt.$$

In section 1, we assumed that we will concern ourselves with only functions of exponential type. In practice, this means that $e^{-st}f(t) \to 0$ as $t \to \infty$. Given this fact, we arrive at

$$\mathscr{L}\{f'(t)\} = -f(0) + s\int_0^\infty f(t)e^{-st}\, dt,$$

or equivalently, the Laplace transform of a derivative is given by

$$\boxed{\mathscr{L}\{f'(t)\} = sF(s) - f(0).} \tag{8}$$

Applying the same process to the second derivative $f''(t)$,

$$\mathscr{L}\{f''(t)\} = s\mathscr{L}\{f'(t)\} - f'(0).$$

Substituting the result for $\mathscr{L}\{f'(t)\}$,

$$\mathscr{L}\{f''(t)\} = s(sF(s) - f(0)) - f'(0),$$

which simplifies to

$$\boxed{\mathscr{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).} \tag{9}$$

By induction, the Laplace transform of the n-th derivative of $f(t)$ is given by

$$\boxed{\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - f^{(n-1)}(0).} \tag{10}$$

This formula shows that each differentiation in the time domain corresponds to multiplication by s in the Laplace domain, along with subtraction of terms involving initial conditions.

This property is fundamental because it transforms differential equations into algebraic equations. For example, if we consider the second order equation

$$y''(t) + p\ y'(t) + q\ y(t) = g(t),$$

we can take the Laplace transform of the left hand side as

$$\begin{aligned}\mathscr{L}\left[y''(t) + p\ y'(t) + q\ y(t)\right] &= \mathscr{L}[y''(t)] + \mathscr{L}[p\ y'(t)] + \mathscr{L}[q\ y(t)] \\ &= \mathscr{L}[y''(t)] + p\mathscr{L}[\ y'(t)] + q\mathscr{L}[\ y(t)] \\ &= s^2Y(s) - sy(0) - y'(0) + p\left(sY(s) - y(0)\right) + qY(s).\end{aligned}$$

Combining with taking the Laplace transform of the right hand side, we arrive at the following *algebraic* equation for the Laplace transform of the solution $Y(s)$ as

$$s^2Y(s) - sy(0) - y'(0) + p(sY(s) - y(0)) + qY(s) = G(s).$$

This is an algebraic equation in $Y(s)$, which can be solved using standard techniques. Once $Y(s)$ is determined, the solution $y(t)$ is obtained by taking the inverse Laplace transform.

3 The inverse Laplace transform

The inverse Laplace transform is the process of recovering a time-domain function $f(t)$ from its Laplace transform $F(s)$. While the Laplace transform converts differential equations into algebraic ones, the inverse transform allows us to return to the original physical or time-domain solution. Formally, the inverse Laplace transform is defined by the complex Bromwich integral:

$$f(t) = \mathscr{L}^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st}F(s)\,ds,$$

where γ is a real constant chosen so that the contour lies to the right of all singularities of $F(s)$.

In practice, this definition is rarely used directly. Instead, inverse transforms are computed using known transform pairs, algebraic manipulation, and decomposition techniques.

3.1 Properties of the inverse Laplace transform

A collection of standard Laplace transform pairs is essential for computing inverse transforms. Some of the most commonly used are:

These basic forms serve as building blocks for more complicated expressions. In practice, we will will manipulate some Laplace transform $F(s)$ to look like one of the transforms

$f(t)$	$\mathscr{L}\{f(t)\} = F(s)$
1	$\frac{1}{s}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}, \quad n \in \mathbb{N}$
e^{at}	$\frac{1}{s-a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$
$e^{at}\sin(\omega t)$	$\frac{\omega}{(s-a)^2+\omega^2}$
$e^{at}\cos(\omega t)$	$\frac{s-a}{(s-a)^2+\omega^2}$
Derivative Properties	
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$
$f^{(n)}(t)$	$s^nF(s) - s^{n-1}f(0) - \cdots - f^{(n-1)}(0)$

above. An important property that allows us to do this comes from the **linearity** of the inverse Laplace transform. This means that for two Laplace transforms $F(s)$ and $G(s)$, we have

$$\mathscr{L}^{-1}\{aF(s) + bG(s)\} = af(t) + bg(t),$$

where $f(t) = \mathscr{L}^{-1}\{F(s)\}$ and $g(t) = \mathscr{L}^{-1}\{G(s)\}$.

3.2 Calculating the inverse Laplace transform

The best way to become proficient at calculating these things is to work through examples. Unfortunately (or fortunately if you enjoy this), this almost always requires **partial fraction decomposition**.

$F(s)$	$\mathscr{L}^{-1}\{F(s)\} = f(t)$
$\frac{1}{s}$	1
$\frac{1}{s^2}$	t
$\frac{n!}{s^{n+1}}$	$t^n, \quad n \in \mathbb{N}$
$\frac{1}{s-a}$	e^{at}
$\frac{\omega}{s^2+\omega^2}$	$\sin(\omega t)$
$\frac{s}{s^2+\omega^2}$	$\cos(\omega t)$
$\frac{\omega}{s^2-\omega^2}$	$\sinh(\omega t)$
$\frac{s}{s^2-\omega^2}$	$\cosh(\omega t)$
$\frac{\omega}{(s-a)^2+\omega^2}$	$e^{at}\sin(\omega t)$
$\frac{s-a}{(s-a)^2+\omega^2}$	$e^{at}\cos(\omega t)$

Table 1: Basic Inverse Laplace Transform Pairs

3.2.1 Review of partial fraction decomposition

Partial fraction decomposition is an algebraic technique used to rewrite a rational function as a sum of simpler fractions. This method is especially useful in computing inverse Laplace transforms, since many transform tables apply directly to simple rational expressions.

The general idea is as follows: suppose we have a rational function

$$\frac{P(s)}{Q(s)},$$

where $P(s)$ and $Q(s)$ are polynomials and $\deg(P) < \deg(Q)$. If $Q(s)$ can be factored into linear or irreducible quadratic factors, then the expression can be written as a sum of simpler fractions whose denominators are these factors. For example, if

$$Q(s) = (s-a)(s-b),$$

then

$$\frac{P(s)}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}.$$

Example 3.1. Find the partial fraction decomposition of

$$\frac{5}{(s+1)(s+3)}.$$

Solution. To find a partial fraction decomposition, we assume that we can write the above rational function as

$$\frac{5}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}.$$

Our goal is to find A and B. To do this, we turn the above equation into a system of equations for A and B. Multiplying both sides by $(s+1)(s+3)$, we have

$$5 = A(s+3) + B(s+1).$$

Expanding, we have

$$5 = s(A+B) + (3A+B)$$

which can be written as the following system:

$$\begin{aligned} 5 &= 3A + B \\ 0 &= A + B. \end{aligned}$$

This system has the solution

$$A = \frac{5}{2} \qquad B = -\frac{5}{2}.$$

Thus, the partial fraction decomposition is

$$\frac{5}{(s+1)(s+3)} = \frac{5/2}{s+1} - \frac{5/2}{s+3}.$$

□

It turns out that for examples like the one above, there is an easier way to compute A and B. This is known as the **cover up method**. When the denominator consists of distinct linear factors, the constants A and B can be found quickly using the cover-up method. Using the example above to illustrate the method. Let's use the above example to illustrate this method. Again, we assume that we can write the rational function as

$$\frac{5}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}.$$

To find A, cover up the factor $(s+1)$ on the left hand side of the above equation and substitute $s = -1$:

$$A = \left.\frac{5}{s+3}\right|_{s=-1} = \frac{5}{2}.$$

To find B, cover up $(s+3)$ on the left hand side of the above equation and substitute $s = -3$:

$$B = \left.\frac{5}{s+1}\right|_{s=-3} = -\frac{5}{2}.$$

We arrive at the same answer as above:

$$\frac{5}{(s+1)(s+3)} = \frac{5/2}{s+1} - \frac{5/2}{s+3}.$$

This method is very useful for calculating partial fraction decompositions quickly. However, beware that this method can **only** be used when the denominator has linear factors. For example, you could **not** use the above method to find A in the decomposition

$$\frac{5}{(s+1)^2(s+4)} = \frac{As+B}{(s+1)^2} + \frac{C}{s+4}.$$

Example 3.2. Find the partial fraction decomposition of

$$\frac{2s+1}{(s-1)(s+2)}.$$

Solution. Using the cover-up method,

$$A = \left.\frac{2s+1}{s+2}\right|_{s=1} = \frac{3}{3} = 1,$$

$$B = \left.\frac{2s+1}{s-1}\right|_{s=-2} = \frac{-3}{-3} = 1.$$

Hence,

$$\frac{2s+1}{(s-1)(s+2)} = \frac{1}{s-1} + \frac{1}{s+2}.$$

□

If the denominator contains repeated linear factors, such as $(s-a)^2$, then the decomposition takes the form

$$\frac{P(s)}{(s-a)^2} = \frac{A}{s-a} + \frac{B}{(s-a)^2}.$$

The cover-up method alone is not sufficient in this case; the coefficients must be found by solving equations or by differentiation. If the denominator contains a quadratic factor that cannot be factored over the reals, such as $s^2+\omega^2$, then the decomposition takes the form

$$\frac{P(s)}{s^2+\omega^2} = \frac{As+B}{s^2+\omega^2}.$$

3.2.2 Examples of finding the inverse Laplace transform

We now explore some examples of calculating the inverse Laplace transform.

Example 3.3. Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} + \frac{2}{s-4}.$$

Solution. Using linearity of the inverse Laplace transform, we have

$$\begin{aligned} f(t) &= \mathscr{L}^{-1}\left(\frac{3}{s} + \frac{2}{s-4}\right) \\ &= \mathscr{L}^{-1}\left(\frac{3}{s}\right) + \mathscr{L}^{-1}\left(\frac{2}{s-4}\right) \\ &= 3\mathscr{L}^{-1}\left(\frac{1}{s}\right) + 2\mathscr{L}^{-1}\left(\frac{1}{s-4}\right) \\ &= 3 + 2e^{4t}. \end{aligned}$$

□

Example 3.4. Find the inverse Laplace transform of

$$F(s) = \frac{5}{s(s+2)}$$

Solution. Computing the partial fraction decomposition of $F(s)$, we find

$$\frac{5}{s(s+2)} = \frac{A}{s} + \frac{B}{s+2}.$$

where we find via the cover up method that $A = \frac{5}{2}$ and $B = -\frac{5}{2}$. We can now easily take the inverse Laplace transform of the partial fraction decomposition as follows:

$$\begin{aligned}
\mathscr{L}^{-1}(F(s)) &= \frac{5}{2}\mathscr{L}^{-1}\left(\frac{1}{s} - \frac{1}{s+2}\right) \\
&= \frac{5}{2}\left(\mathscr{L}^{-1}\left(\frac{1}{s}\right) - \mathscr{L}^{-1}\left(\frac{1}{s+2}\right)\right) \\
&= \frac{5}{2}\left(1 - e^{-2t}\right)
\end{aligned}$$

□

Example 3.5. Find the inverse Laplace transform of

$$F(s) = \frac{2s+3}{s^2+4s+5}$$

Solution. The denominator of the above transfer function can be simplified by completing the square

$$s^2 + 4s + 5 = (s+2)^2 + 1$$

and the numerator as

$$2s + 3 = 2(s+2) - 1.$$

This yields

$$\frac{2s+3}{s^2+4s+5} = \frac{2(s+2)}{(s+2)^2+1} - \frac{1}{(s+2)^2+1}.$$

Here, we need to make an important observation. Note that the above partial fraction decomposition can be written in the following way: If $G(s) = \frac{2s}{s^2+1} - \frac{1}{s^2+1}$, then we can calculate the inverse Laplace transform as

$$\begin{aligned}
\mathscr{L}^{-1}(G(s)) &= \mathscr{L}^{-1}\left(\frac{2s}{s^2+1} - \frac{1}{s^2+1}\right) \\
&= 2\,\mathscr{L}^{-1}\left(\frac{s}{s^2+1}\right) - \mathscr{L}^{-1}\left(\frac{1}{s^2+1}\right) \\
&= 2\cos(t) - \sin(t).
\end{aligned}$$

Now, want to find the inverse Laplace transform of $G(s+2)$. Using the exponential shift law (3), we have

$$\mathscr{L}(e^{at}f(t)) = F(s-a), \quad s > a.$$

Therefore, the inverse Laplace transform of $G(s+2)$ is given by

$$e^{-2t}\mathscr{L}^{-1}(G(s)) = 2e^{-2t}\cos(t) - e^{-2t}\sin(t)$$

□

The above example yields the following proposition.

Proposition 3.6. *Suppose that function $f(t)$ has the Laplace transform*

$$\mathscr{L}\{f(t)\} = F(s),$$

then

$$\boxed{\mathscr{L}^{-1}\{F(s-a)\} = e^{at}f(t)}.$$

4 Solving differential equations with the Laplace transform

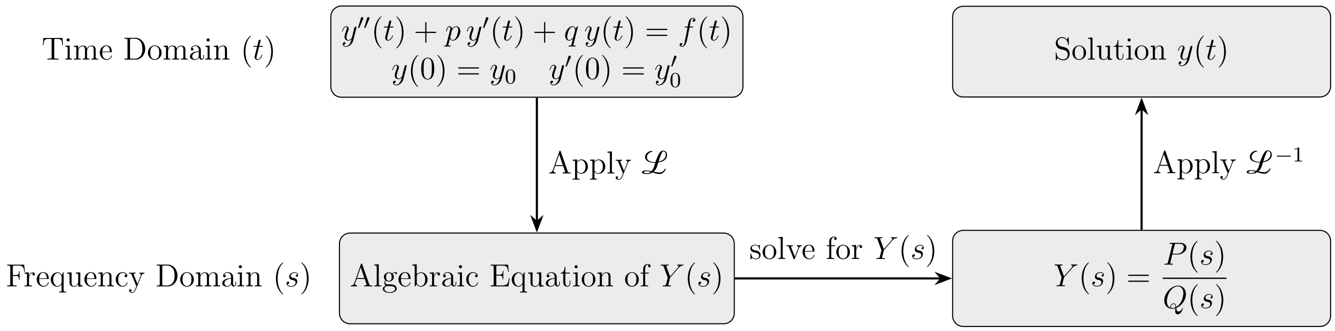
One of the most powerful applications of the Laplace transform is the solution of linear ordinary differential equations (ODEs), particularly initial value problems. The key advantage of this method is that it converts differential equations in the time domain into algebraic equations in the complex frequency domain, which are typically much easier to solve.

4.1 The general procedure for solving ODEs with the Laplace transform

To solve a differential equation using the Laplace transform, the following steps are followed:

1. Take the Laplace transform of both sides of the differential equation.
2. Use the derivative properties to replace derivatives with algebraic expressions involving s and initial conditions.
3. Solve the resulting algebraic equation for $F(s)$, the Laplace transform of the unknown function $f(t)$.
4. Decompose $F(s)$ into simpler terms using algebraic manipulation (often partial fractions).
5. Take the inverse Laplace transform to obtain the solution $f(t)$.

Consider the following diagram that depicts how we solve differential equations using the Laplace transform.



4.2 First-order differential equations

Example 4.1. Solve the initial value problem

$$y'(t) + 2y(t) = 3, \quad y(0) = 1.$$

Solution. Taking the Laplace transform of both sides:

$$\mathscr{L}\{y'(t)\} + 2\mathscr{L}\{y(t)\} = \mathscr{L}\{3\}.$$

Using the Laplace transform of the first derivative formula in (8), we have

$$(sY(s) - y(0)) + 2Y(s) = \frac{3}{s}.$$

Substituting the initial condition $y(0) = 1$ yields

$$sY(s) - 1 + 2Y(s) = \frac{3}{s}.$$

Solving for $Y(s)$, we arrive at

$$Y(s) = \frac{3}{s(s+2)} + \frac{1}{s+2}.$$

As the first expression has no standard inverse Laplace transform, we need to find the partial fraction decomposition. Using the cover up method, we calculate:

$$\frac{3}{s(s+2)} = \frac{3/2}{s} - \frac{3/2}{s+2}$$

and hence

$$F(s) = \frac{3}{2}\frac{1}{s} - \frac{1}{2}\frac{1}{s+2}.$$

We can now take the inverse Laplace transform of both terms to find

$$y(t) = \frac{3}{2} - \frac{1}{2}e^{-2t}$$

□

Example 4.2. Solve the initial value problem

$$y'(t) + y(t) = \cos t, \quad y(0) = 0.$$

Solution. Taking the Laplace transform of both sides:

$$\mathscr{L}\{y'(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{\cos t\}.$$

Using the Laplace transform of the first derivative formula in (8) and the Laplace transform of $\sin(at)$ in (6), we have

$$(sY(s) - y(0)) + Y(s) = \frac{s}{s^2+1}.$$

Substituting the initial condition $y(0) = 0$ yields

$$(s+1)Y(s) = \frac{s}{s^2+1}.$$

Solving for $Y(s)$, we arrive at

$$Y(s) = \frac{s}{(s+1)(s^2+1)}.$$

As the right hand side has no standard inverse Laplace transform, we need to find the partial fraction decomposition. Unfortunately, the cover up method will *not* work in this case for the terms $\frac{Bs+C}{s^2+1}$ (because the denominator cannot be factored into linear factors over the integers). You could, however, factor the denominator over the complex numbers, yielding linear factors of $s+i$ and $s-i$. I try to avoid this method because it will involve using the complex exponential formulae for $\cos(at)$ and $\sin(at)$– adding an extra step. However, if you want to get more practice, you are welcome to use this method, too! Anyways, here we will find the partial fraction decomposition the *old fashion way*. First, we have

$$\frac{s}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}.$$

Multiplying through, expanding, and combining like terms of constants, s, and s^2, we arrive at

$$\begin{aligned} s &= A(s^2+1) + (Bs+C)(s+1) \\ s &= As^2 + A + Bs^2 + Bs + Cs + C \\ s &= (A+B)s^2 + (B+C)s + (A+C) \end{aligned}$$

We can match coefficients on the left and right hand sides which yields the following system for A B, and C:

$$A + B = 0, \quad B + C = 1, \quad A + C = 0.$$

We can solve this system by standard substitution methods to find

$$A = -\frac{1}{2}, \quad B = \frac{1}{2}, \quad C = \frac{1}{2}.$$

Hence, the $Y(s)$ has the partial fraction decomposition of

$$Y(s) = -\frac{1}{2}\frac{1}{s+1} + \frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1}.$$

We can now take the inverse Laplace transform of the above equation to find the solution $y(t)$ to be

$$y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}\cos t + \frac{1}{2}\sin t.$$

□

4.3 Second-order differential equations

Example 4.3. Solve the initial value problem

$$y''(t) + 3y'(t) + 2y(t) = 0, \quad y(0) = 0, \quad y'(0) = 1.$$

Solution. Taking Laplace transforms:

$$\mathscr{L}\{f''(t)\} + 3\mathscr{L}\{f'(t)\} + 2\mathscr{L}\{f(t)\} = 0.$$

Using the Laplace transform of the first derivative formula in (8) and the Laplace transform of the second derivative formula in (9), we have

$$(s^2Y(s) - sy(0) - y'(0)) + 3(sY(s) - y(0)) + 2Y(s) = 0.$$

Substituting the initial conditions $y(0) = 0$ and $y'(0) = 1$ yields

$$(s^2Y(s) - 1) + 3(sY(s)) + 2Y(s) = 0.$$

Solving for $Y(s)$, we arrive at

$$Y(s) = \frac{1}{(s+1)(s+2)}.$$

Again, to calculate the inverse Laplace transform of $Y(s)$, partial fraction decomposition is required. Luckily, this is a decomposition that can use the cover up method. We have

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}.$$

We have that

$$A = \lim_{s\to-2}\frac{1}{s+1} = -1$$
$$B = \lim_{s\to-1}\frac{1}{s+2} = 1$$

and hence

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2}.$$

We can now take the inverse Laplace transform of the above equation to find the solution $y(t)$ to be

$$y(t) = e^{-t} - e^{-2t}.$$

□

Example 4.4. Solve the initial value problem

$$y''(t) + y(t) = \sin t, \quad y(0) = 0, \quad y'(0) = 0.$$

Solution. Taking Laplace transforms:

$$(s^2 F(s)) + F(s) = \frac{1}{s^2+1}.$$

$$(s^2+1)F(s) = \frac{1}{s^2+1}.$$

$$F(s) = \frac{1}{(s^2+1)^2}.$$

Using a known inverse transform:

$$\mathcal{L}^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} = \frac{1}{2}(\sin t - t\cos t).$$

Thus,

$$f(t) = \frac{1}{2}(\sin t - t\cos t).$$

□

5 Piecewise input functions

Up to this point, we have focused on solving differential equations with smooth, continuous forcing functions. While these examples are important for understanding the mechanics of the Laplace transform, they do not fully capture why this method is so powerful. In real-world applications, inputs are rarely perfectly smooth. Instead, systems are often subject to forces that turn on or off at specific times, signals that are applied for only a finite duration, or behaviors that change abruptly.

These types of inputs are naturally described by *piecewise functions*, which are typically difficult to handle using standard solution techniques. Classical methods require breaking the problem into multiple intervals, solving separate differential equations on each interval, and carefully matching conditions at every transition point. As the number of discontinuities increases, this process becomes increasingly complicated and inefficient.

The Laplace transform provides a fundamentally different approach. Rather than treating each interval separately, it allows us to encode all of this behavior into a single expression using the unit step function. Discontinuities and time delays are transformed into simple algebraic factors, and the entire problem can be solved at once in the s-domain. When we return to the time domain, these features reappear naturally in the form of shifted functions multiplied by step functions.

This ability to handle discontinuous and delayed inputs is one of the primary reasons the Laplace transform is so widely used. It is not merely a convenient alternative to other methods—it is specifically designed to model systems that change over time in abrupt or non-smooth ways. In many practical situations, it is the most natural and efficient tool available.

5.1 The unit step function

The unit step function is one of the most important tools for working with discontinuous functions in the Laplace transform framework. It allows us to rewrite piecewise-defined functions as single algebraic expressions, which can then be transformed and manipulated systematically.

The **unit step function** (also called the Heaviside function) is defined by

$$u(t-a) = \begin{cases} 0, & t < a, \\ 1, & t \geq a. \end{cases}$$

This function "turns on" at $t = a$. Before $t = a$, it is zero; after $t = a$, it is one. Because of this behavior, it is often used to model signals or inputs that begin at a specific time. The unit step function is given pictorally below. m

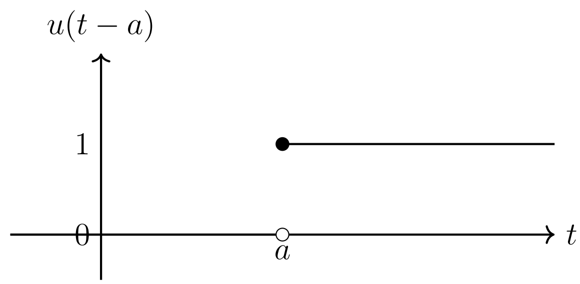


Figure 1: Unit step function $u(t-a)$

The reason why we are interested in the unit step function is that if we multiply a function $f(t)$ by $u(t-a)$, the function will take the value zero until $t = a$, at which point the function will assume its defined value.

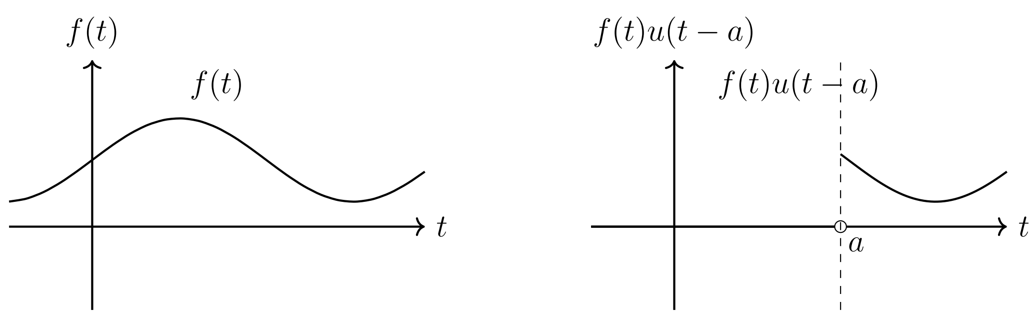


Figure 2: Effect of multiplying by the unit step function: the function is "turned on" at $t = a$

Suppose instead that we want to shift the entire function, not only turn different parts of the function on and off. The unit step function can handle this, too. Instead of multiplying $u(t-a)$ to $f(t)$, we should multiply $u(t-a)$ by $f(t-a)$. This is shown pictorially below.

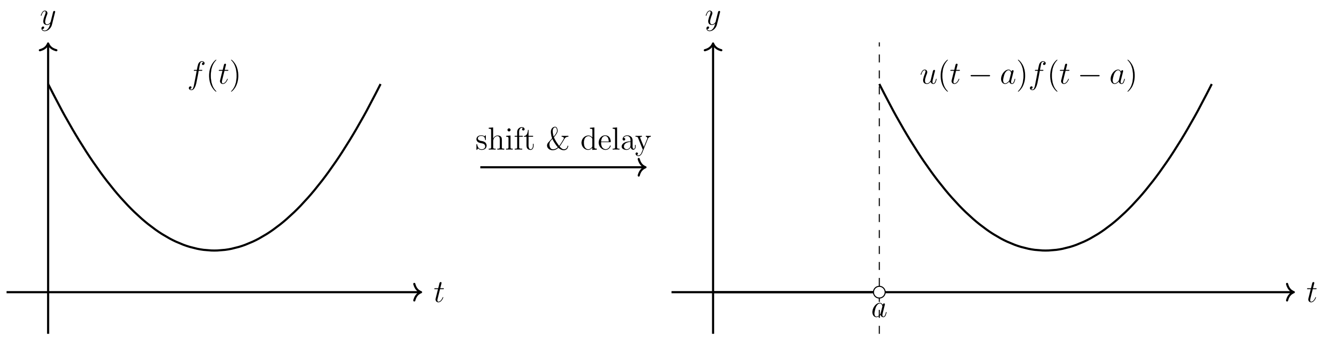


Figure 3: A shifted (delayed) function: multiplication by $u(t-a)$ both shifts the function and forces it to be zero for $t < a$

Example 5.1. Write the function

$$f(t) = \begin{cases} 0, & t < 2, \\ t-2, & t \geq 2, \end{cases}$$

in terms of the unit step function.

Solution. Since the function begins at $t = 2$ and looks like $t - 2$, we write

$$f(t) = u(t-2)(t-2).$$

□

Suppose that instead of turning a certain function on, we wanted to turn a function off. We illustrate this in the following example.

Example 5.2. Write the function

$$f(t) = \begin{cases} 1, & 1 \leq t < 3, \\ 0, & \text{otherwise}, \end{cases}$$

in terms of step functions.

Solution. This function turns on at $t = 1$ and off at $t = 3$, so we write

$$f(t) = u(t-1) - u(t-3).$$

□

For more complicated piecewise functions, we build the expression step by step by adding the changes that occur at each transition point.

Example 5.3. Write

$$f(t) = \begin{cases} t, & 0 \leq t < 2, \\ 3, & t \geq 2, \end{cases}$$

using step functions.

Solution. Start with the initial behavior t, then correct it at $t = 2$. At that point, the function changes from t to 3, so we subtract the old behavior and add the new one:

$$f(t) = t + u(t-2)\big(3-t\big).$$

□

Example 5.4. Write

$$f(t) = \begin{cases} 0, & t < 1, \\ t-1, & 1 \le t < 3, \\ 2, & t \ge 3, \end{cases}$$

using unit step functions.

Solution. We build this in stages:

- At $t = 1$, the function becomes $t - 1$: add $u(t-1)(t-1)$.
- At $t = 3$, the function changes from $t - 1$ to 2: add a correction term.

Thus,

$$f(t) = u(t-1)(t-1) + u(t-3)\big(2-(t-1)\big).$$

□

5.2 The Laplace transform of the Unit Step Function

To calculate the Laplace transform of $u(t-a)$, we substitute into the definition of Laplace transform in (1) to find

$$\mathscr{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a)\, dt.$$

Since $u(t-a) = 0$ for $t < a$ and $u(t-a) = 1$ for $t \ge a$, the integral simplifies to

$$\mathscr{L}\{u(t-a)\} = \int_a^\infty e^{-st}\, dt.$$

We can compute the above integral as

$$\begin{aligned} \int_a^\infty e^{-st}\, dt &= \left[-\frac{1}{s}e^{-st}\right]_{t=a}^\infty \\ &= 0 - \left(-\frac{1}{s}e^{-as}\right) \\ &= \frac{e^{-as}}{s}. \end{aligned}$$

where we assume that $\text{Re}(s) > 0$, the exponential term vanishes as $t \to \infty$. Thus, we obtain the fundamental result:

$$\boxed{\mathscr{L}\{u(t-a)\} = \frac{e^{-as}}{s}}$$

This result is extremely important. It shows the following important connection between the shifting in the spatial domain and exponentiation in the frequency domain.

- A **delay in time** (the function turning on at $t = a$) becomes
- A **multiplicative exponential factor** e^{-as} in the s-domain.

In other words, the Laplace transform converts the timing of an event into algebra. Notice that when $a = 0$, we recover the basic transform:

$$\mathscr{L}\{u(t)\} = \frac{1}{s}.$$

Thus, $u(t-a)$ can be viewed as a shifted version of the constant function 1. The exponential factor e^{-as} precisely encodes how far the function has shifted. This formula is the foundation for handling discontinuous and delayed inputs. Any function that turns on at time $t = a$ can be written in the form

$$u(t-a)f(t-a),$$

and its Laplace transform will include the factor e^{-as}.

This means:

- The *shape* of the function is handled by $F(s)$,
- The *timing* of the function is handled by e^{-as}.

This separation of shape and timing is one of the key reasons the Laplace transform is so effective for solving problems involving delayed or piecewise inputs.

5.3 The uniqueness of the inverse Laplace transform

An important theoretical property of the Laplace transform is the **uniqueness of the inverse**. This guarantees that when we compute an inverse Laplace transform, the function we obtain is the *only* function (under reasonable conditions) that corresponds to the given transform.

If two functions $f(t)$ and $g(t)$ have the same Laplace transform, that is,

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{g(t)\},$$

then

$$f(t) = g(t)$$

for all $t \geq 0$, except possibly at isolated points of discontinuity. This result is crucial because it ensures that the process of solving for $F(s)$ and then taking the inverse Laplace transform produces a *well-defined and unique solution* to a differential equation.

It is important to note that the Laplace transform only depends on the values of a function for $t \geq 0$. This means:

- Two functions that differ for $t < 0$ will still have the same Laplace transform.
- The inverse Laplace transform only recovers the function on $[0, \infty)$.

For this reason, Laplace transforms are always interpreted in the context of **causal functions**, where we assume

$$f(t) = 0 \quad \text{for } t < 0.$$

Because the Laplace transform ignores negative time, every inverse Laplace transform should be understood as a *causal function*. In practice, this means we should write results in the form

$$f(t)\, u(t),$$

even if the factor $u(t)$ is not always written explicitly. More generally, when delays are involved, inverse transforms naturally take the form

$$u(t-a)f(t-a).$$

This ensures that:

- The function is zero before $t = a$,
- The function behaves like $f(t-a)$ after $t = a$,
- The solution is consistent with the one-sided nature of the Laplace transform.

5.4 The second shifting theorem

We now present a very useful formula for calculating inverse Laplace transforms for transfer function that include a transfer function.

Theorem 5.5. *Suppose*

$$\mathscr{L}\{f(t)\} = F(s).$$

Then for $a > 0$,

$$\boxed{\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}F(s)}. \tag{11}$$

Proof. From the definition of the Laplace transform

$$\mathscr{L}\{u(t-a)f(t-a)\} = \int_0^\infty e^{-st}u(t-a)f(t-a)\, dt.$$

Since $u(t-a) = 0$ for $t < a$ and $u(t-a) = 1$ for $t \geq a$, the integral becomes

$$\mathscr{L}\{u(t-a)f(t-a)\} = \int_a^\infty e^{-st}f(t-a)\, dt.$$

Now make the substitution

$$\tau = t - a, \qquad t = \tau + a, \qquad dt = d\tau.$$

When $t = a$, we have $\tau = 0$, and when $t \to \infty$, we have $\tau \to \infty$. Thus

$$\mathscr{L}\{u(t-a)f(t-a)\} = \int_0^\infty e^{-s(\tau+a)}f(\tau)\, d\tau.$$

Factor out the term e^{-as}:

$$\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)\, d\tau.$$

Recognizing the remaining integral as $F(s)$, we obtain

$$\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}F(s).$$

□

Since

$$\mathscr{L}\{u(t-a)f(t-a)\} = e^{-as}F(s),$$

the inverse Laplace transform can be written as

$$\boxed{\mathscr{L}^{-1}\{e^{-as}F(s)\} = u(t-a)f(t-a).} \tag{12}$$

This is the form most often used when solving differential equations with delayed forcing functions.

5.5 Solving differential equations with a piecewise input

Example 5.6. Solve the initial value problem

$$y'(t) + y(t) = f(t), \qquad y(0) = 0,$$

where the input is piecewise-defined by

$$f(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1, & t \ge 1. \end{cases}$$

Solution. This is exactly the kind of forcing term that the Laplace transform handles well. We first rewrite the input using the unit step function:

$$f(t) = u(t-1).$$

So, the differential equation becomes

$$y'(t) + y(t) = u(t-1), \qquad y(0) = 0.$$

Applying the Laplace transform gives

$$\mathscr{L}\{y'(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{u(t-1)\}.$$

Using formula (8) which states the Laplace transform of the first derivative and substituting the initial condition that $y(0)0$, we have

$$sY(s) + Y(s) = \mathscr{L}\{u(t-1)\}.$$

Calculating the Laplace transform of the right hand side, we have

$$\mathscr{L}\{u(t-1)\} = \frac{e^{-s}}{s}$$

and hence

$$Y(s) = \frac{e^{-s}}{s(s+1)}.$$

When solving these types of problem, it is wise to separate the exponential and rational parts of the transfer function in the following way. This allows us to now easily calculate the partial fraction decomposition of the rational portion.

$$Y(s) = e^{-s}\left(\frac{1}{s(s+1)}\right).$$

The partial fraction decomposition of the rational part is given by

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}.$$

and hence

$$1 = A(s+1) + Bs.$$

We can easily see that

$$A = 1$$
$$B = -1$$

which yields the following partial fraction decomposition

$$\frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1}.$$

Now, we appeal to equation (12) which provides that

$$\mathscr{L}^{-1}(e^{-as}F(s)) = u(t-a)f(t-a).$$

Accordingly, we have to calculate the inverse Laplace transform of the rational part of the above transfer function:

$$\frac{1}{s} - \frac{1}{s+1}$$

which has the inverse Laplace transform of

$$1 - e^{-t}.$$

Finally, applying equation (12), we arrive at the solution

$$y(t) = u(t-1)\left(1 - e^{-(t-1)}\right).$$

If we want to write the solution as a piecewise function, we find that

$$y(t) = \begin{cases} 0, & 0 \le t < 1, \\ 1 - e^{-(t-1)}, & t \ge 1. \end{cases}$$

□

Example 5.7. Solve the initial value problem

$$y''(t) + 3y'(t) + 2y(t) = f(t), \qquad y(0) = 0, \qquad y'(0) = 0,$$

where the forcing function is piecewise-defined by

$$f(t) = \begin{cases} 0, & 0 \leq t < 1, \\ 1, & 1 \leq t < 3, \\ 0, & t \geq 3. \end{cases}$$

Solution. The first step is to rewrite the piecewise input using the unit step function. Since the function turns on at $t = 1$ and turns off at $t = 3$, we can write

$$f(t) = u(t-1) - u(t-3).$$

So the differential equation becomes

$$y''(t) + 3y'(t) + 2y(t) = u(t-1) - u(t-3).$$

Applying the Laplace transform gives

$$\mathscr{L}\{y''(t)\} + 3\mathscr{L}\{y'(t)\} + 2\mathscr{L}\{y(t)\} = \mathscr{L}\{u(t-1)\} - \mathscr{L}\{u(t-3)\}.$$

Using the Laplace transform derivative formulas (8) and (9) and the initial conditions $y(0) = 0$ and $y'(0) = 0$, we get

$$\big(s^2Y(s) - sy(0) - y'(0)\big) + 3\left(sY(s) - y(0)\right) + 2Y(s) = \frac{e^{-s}}{s} - \frac{e^{-3s}}{s}.$$

Since the initial conditions are zero, this simplifies to

$$s^2Y(s) + 3sY(s) + 2Y(s) = \frac{e^{-s}}{s} - \frac{e^{-3s}}{s}.$$

Solving for $Y(s)$ gives

$$Y(s) = \frac{e^{-s} - e^{-3s}}{s(s+1)(s+2)}.$$

Separate the exponential term from the rational terms

$$Y(s) = e^{-s}\frac{1}{s(s+1)(s+2)} - e^{-3s}\frac{1}{s(s+1)(s+2)}.$$

From (12), we will need to calculate the Laplace transform of the rational part above. It is a good idea to define

$$G(s) = \frac{1}{s(s+1)(s+2)}$$

and hence

$$Y(s) = e^{-s}G(s) - e^{-3s}G(s).$$

We now decompose $G(s)$ into partial fractions

$$\frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}.$$

Multiplying through by $s(s+1)(s+2)$ gives

$$1 = A(s+1)(s+2) + Bs(s+2) + Cs(s+1).$$

Another method to find the coefficient is presented as follows. We can pick connivent choices of s that will easily give the coefficients. First, if $s = 0$, we have

$$1 = A(1)(2) = 2A,$$

so

$$A = \frac{1}{2}.$$

Next, if $s = -1$, then

$$1 = B(-1)(1) = -B,$$

so

$$B = -1.$$

Finally, if $s = -2$, then

$$1 = C(-2)(-1) = 2C,$$

so

$$C = \frac{1}{2}.$$

Therefore,

$$G(s) = \frac{1}{2s} - \frac{1}{s+1} + \frac{1}{2(s+2)}.$$

Taking the inverse Laplace transform,

$$g(t) = \mathscr{L}^{-1}\{G(s)\} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}.$$

Since

$$Y(s) = e^{-s}G(s) - e^{-3s}G(s),$$

the second shifting theorem gives

$$y(t) = u(t-1)g(t-1) - u(t-3)g(t-3).$$

Substituting the formula for $g(t)$,

$$y(t) = u(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) - u(t-3)\left(\frac{1}{2} - e^{-(t-3)} + \frac{1}{2}e^{-2(t-3)}\right).$$

So the solution in unit step form is

$$y(t) = u(t-1)\left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) - u(t-3)\left(\frac{1}{2} - e^{-(t-3)} + \frac{1}{2}e^{-2(t-3)}\right)$$

Because the input turns on at $t = 1$ and turns off at $t = 3$, the solution naturally has three time intervals.

$$y(t) = \begin{cases} 0, & 0 \le t < 1, \\ \frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}, & 1 \le t < 3, \\ \left(\frac{1}{2} - e^{-(t-1)} + \frac{1}{2}e^{-2(t-1)}\right) - \left(\frac{1}{2} - e^{-(t-3)} + \frac{1}{2}e^{-2(t-3)}\right), & t \ge 3. \end{cases}$$

□

6 The Delta distribution

The Dirac delta distribution, denoted $\delta(t-a)$, is a mathematical object used to model an instantaneous impulse occurring at time $t = a$. It is not a function in the traditional sense, but rather a **distribution** (or generalized function), defined by how it behaves under integration. The delta distribution plays a central role in Laplace transforms, especially when modeling systems subject to sudden forces or inputs.

6.1 Basic definitions and properties

The defining property of the delta distribution is the *sifting property*:

$$\int_{-\infty}^{\infty} \delta(t-a) f(t)\, dt = f(a).$$

This means that $\delta(t-a)$ *picks out* the value of a function at $t = a$. In other words, the delta distribution *turns on* at a single point. It has the following two important properties:

1. $\delta(t-a) = 0 \quad \text{for } t \ne a$
2. $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$

The delta distribution is typically represented as an arrow at $t = a$ with unit area.

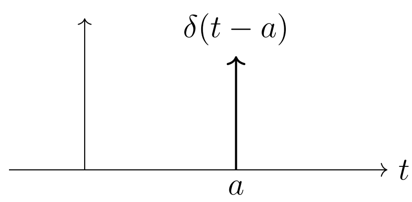


Figure 4: Dirac delta distribution

The delta distribution has a nice relation to the unit step function we studied in the previous chapter. If we consider the unit box $f(t) = u(t) - u(t-1)$ and make the right end point as a parameter, say $\dfrac{1}{h}$ while keeping unit area, we have the box given by

$$f(t) = h\left(u(t) - u\left(t - \frac{1}{h}\right)\right)$$

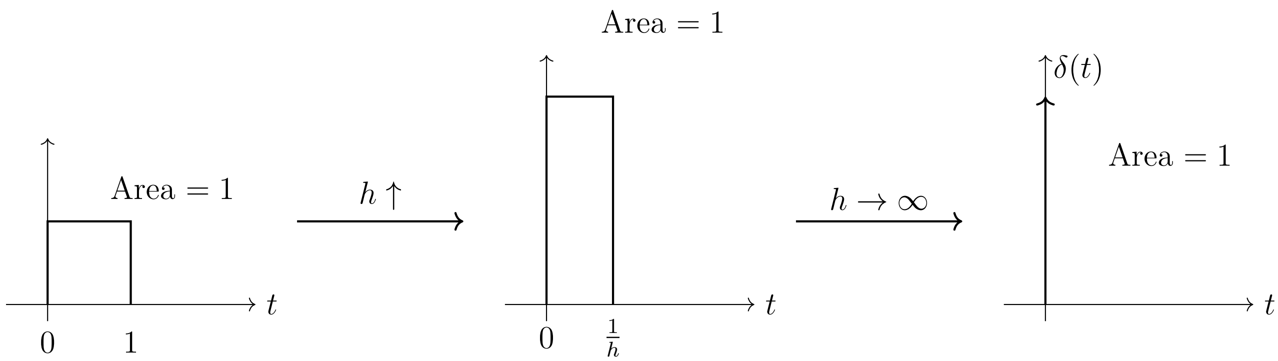


Figure 5: A sequence of functions with width $\frac{1}{h}$ and height h approaching the delta distribution as $h \to \infty$. The area remains equal to 1.

$$\delta(t) = \lim_{h\to\infty} \begin{cases} h, & 0 \le t \le \frac{1}{h}, \\ 0, & \text{otherwise.} \end{cases}$$

The delta distribution is the derivative of the unit step function:

$$\frac{d}{dt}u(t-a) = \delta(t-a).$$

This means that the delta distribution represents an instantaneous jump in the step function. In this sense, the step function accumulates the delta, and the delta measures how the step changes. The delta distribution is not a classical function because it is zero everywhere except at a single point, yet integrates to 1. Instead, it is defined through its action on other functions (via integration). Objects defined in this way are called **distributions**. They allow us to rigorously describe idealized impulses that cannot be represented by ordinary functions. The delta distribution models instantaneous effects such as:

- A hammer strike applying a sudden force,
- A voltage spike in an electrical circuit,
- A sudden injection of mass or charge,
- An instantaneous impulse applied to a mechanical system.

In each case, the effect occurs over an extremely short time but has a finite total impact.

6.2 The Laplace transform of the delta distribution

One way to understand the Laplace transform of the delta distribution is to view the delta as the limit of a very narrow rectangle whose area is always equal to 1. This is often called a *unit box approximation.* Define

$$\delta_h(t) = \begin{cases} h, & 0 \le t \le \dfrac{1}{h}, \\ 0, & \text{otherwise.} \end{cases}$$

This function has height h, width $\frac{1}{h}$, and therefore area

$$h \cdot \frac{1}{h} = 1.$$

As $h \to \infty$, the box becomes narrower and taller, and in the limit it approaches the delta distribution $\delta(t)$. Now compute the Laplace transform of $\delta_h(t)$:

$$\mathscr{L}\{\delta_h(t)\} = \int_0^\infty e^{-st}\delta_h(t)\,dt.$$

Since $\delta_h(t)$ is only nonzero on $[0, 1/h]$, this becomes

$$\mathscr{L}\{\delta_h(t)\} = \int_0^{1/h} he^{-st}\,dt.$$

Evaluating the integral,

$$\mathscr{L}\{\delta_h(t)\} = h\left[\frac{-1}{s}e^{-st}\right]_0^{1/h} = \frac{h}{s}\left(1 - e^{-s/h}\right).$$

Now take the limit as $h \to \infty$:

$$\mathscr{L}\{\delta(t)\} = \lim_{h\to\infty}\mathscr{L}\{\delta_h(t)\} = \lim_{h\to\infty}\frac{h}{s}\left(1 - e^{-s/h}\right).$$

To evaluate the limit, use the series expansion for the exponential to see

$$e^{-s/h} = 1 - \frac{s}{h} + O\left(\frac{1}{h^2}\right),$$

so

$$1 - e^{-s/h} = \frac{s}{h} + O\left(\frac{1}{h^2}\right).$$

Substituting into the expression above gives

$$\frac{h}{s}\left(1 - e^{-s/h}\right) \to \frac{h}{s}\cdot\frac{s}{h} = 1.$$

Therefore,

$$\boxed{\mathscr{L}\{\delta(t)\} = 1.} \tag{13}$$

The same idea works for a delta located at $t = a$. Define

$$\delta_h(t-a) = \begin{cases} h, & a \le t \le a + \dfrac{1}{h}, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\mathscr{L}\{\delta_h(t-a)\} = \int_a^{a+1/h} he^{-st}\,dt.$$

Evaluating,

$$\mathscr{L}\{\delta_h(t-a)\} = h\left[\frac{-1}{s}e^{-st}\right]_a^{a+1/h} = \frac{h}{s}\left(e^{-as} - e^{-s(a+1/h)}\right).$$

Factor out e^{-as}:

$$\mathscr{L}\{\delta_h(t-a)\} = e^{-as}\frac{h}{s}\left(1 - e^{-s/h}\right).$$

Now, let $h \to \infty$:

$$\mathscr{L}\{\delta(t-a)\} = e^{-as}\lim_{h\to\infty}\frac{h}{s}\left(1 - e^{-s/h}\right) = e^{-as}.$$

Hence,

$$\boxed{\mathscr{L}\{\delta(t-a)\} = e^{-as}.} \tag{14}$$

This derivation shows that the delta distribution can be understood as the limit of a sequence of unit-area box functions. As the box becomes infinitely narrow and infinitely tall, its Laplace transform approaches the simple exponential factor

$$e^{-as}$$

which is why impulses are so easy to represent in the Laplace domain.

6.3 Solving Differential Equations with Delta Inputs

Differential equations with delta inputs describe systems subjected to instantaneous impulses. These impulses often produce sudden changes in the solution or its derivatives.

Example 6.1. Solve the initial value problem

$$y'(t) + y(t) = \delta(t-2), \qquad y(0) = 0.$$

Solution. Applying the Laplace transform to both sides and using (8) yields

$$(sY(s) - y(0)) + Y(s) = e^{-2s},$$

and as $y(0) = 0$

$$(s+1)Y(s) = e^{-2s}.$$

We can isolate $Y(s)$ as

$$Y(s) = \frac{e^{-2s}}{s+1}.$$

Now take the inverse Laplace transform. First,

$$\mathscr{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}.$$

Applying the shifting theorem,

$$y(t) = u(t-2)e^{-(t-2)}.$$

Thus,

$$y(t) = \begin{cases} 0, & t < 2, \\ e^{-(t-2)}, & t \geq 2. \end{cases}$$

□

Example 6.2. Solve

$$y''(t) + y(t) = \delta(t - \pi), \qquad y(0) = 0, \quad y'(0) = 0.$$

Solution. Take the Laplace transform of both sides and apply (9) and (14) with applying the initial conditions yields

$$s^2 Y(s) + Y(s) = e^{-\pi s}.$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{e^{-\pi s}}{s^2 + 1}.$$

As we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 + 1}\right\} = \sin t,$$

we can apply (17) which yields

$$y(t) = u(t - \pi)\sin(t - \pi).$$

Writing the solution as a piecewise function yields

$$y(t) = \begin{cases} 0, & t < \pi, \\ \sin(t - \pi), & t \geq \pi. \end{cases}$$

□

7 The Convolution

When working with Laplace transforms, one quickly becomes accustomed to the elegance with which certain operations in the time domain translate into simpler operations in the transform domain. For instance, differentiation and integration correspond to multiplication and division by the complex variable s, respectively. This naturally raises the question: what happens to the *product* of two functions?

More precisely, given two functions $f(t)$ and $g(t)$, one might hope for a simple relationship between the Laplace transform of their product $f(t)g(t)$ and the individual transforms $F(s)$ and $G(s)$. Such a formula would be highly desirable, as products of functions arise frequently in applications, including modulation, weighting, and nonlinear interactions.

However, unlike differentiation or convolution, there is no straightforward formula expressing $\mathcal{L}\{f(t)g(t)\}$ purely in terms of $F(s)$ and $G(s)$. In this sense, the Laplace transform does not interact cleanly with pointwise multiplication.

Instead, the situation is reversed: a remarkably simple relationship exists for a different operation, known as *convolution*. The convolution of two functions f and g, defined by

$$\boxed{(f * g)(t) = \int_0^t f(\tau)\, g(t - \tau)\, d\tau,} \tag{15}$$

plays a central role in the theory. While it may initially appear more complicated than pointwise multiplication, it is precisely this operation that transforms neatly under the Laplace transform.

In fact, as we will soon see, the Laplace transform converts convolution in the time domain into multiplication in the transform domain. Thus, although we lack a simple formula for the transform of a product, convolution provides the closest and most useful analogue—one that underpins many applications in differential equations, systems theory, and signal processing.

7.1 Basic calculations

In many differential equation problems, we eventually arrive at a Laplace-domain expression of the form

$$F(s)G(s),$$

where each factor has a simple inverse Laplace transform, but the product itself is not easy to rewrite using standard tables. In such situations, convolution provides a systematic way to return to the time domain.

The idea is that instead of multiplying two functions in the s-domain and trying to invert the result directly, we can interpret the product as the Laplace transform of a convolution in the t-domain. This is especially useful when:

- partial fractions are difficult or impossible to use cleanly,
- the forcing term is a product of transforms,
- the differential equation produces a response that is naturally written as an integral,
- one wants to describe how one signal is "smeared" or "spread out" by another.

Convolution also appears in applications such as systems theory, probability, and physics. In a linear system, the output produced by an input can often be viewed as a convolution of the input with the system's impulse response. This makes convolution a fundamental operation rather than just a computational trick.

Here are two standard examples that show how the convolution integral is computed in practice.

Example 7.1. Calculate the convolution between the functions $f(t) = t$ and $g(t) = 1$.

Solution. Using the definition (15), we need to evaluate the following integral

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

So

$$(1 * t)(t) = \int_0^t 1 \cdot (t-\tau)\, d\tau = \int_0^t (t-\tau)\, d\tau.$$

Evaluating the integral gives

$$\begin{aligned}\int_0^t (t-\tau)\,d\tau &= \left[t\tau - \frac{\tau^2}{2}\right]_0^t \\ &= t^2 - \frac{t^2}{2} \\ &= \frac{t^2}{2}.\end{aligned}$$

Hence,

$$(1 * t)(t) = \frac{t^2}{2}.$$

□

Example 7.2. Calculate the convolution between the functions $f(t) = e^{at}$ and $g(t) = e^{bt}$.

Solution. Again using definition (15),

$$(e^{at} * e^{bt})(t) = \int_0^t e^{a\tau} e^{b(t-\tau)}\,d\tau.$$

Pulling out the terms that do not depend on τ,

$$(e^{at} * e^{bt})(t) = e^{bt} \int_0^t e^{(a-b)\tau}\,d\tau.$$

If $a \neq b$, then

$$\int_0^t e^{(a-b)\tau}\,d\tau = \frac{e^{(a-b)t} - 1}{a-b},$$

so

$$(e^{at} * e^{bt})(t) = \frac{e^{at} - e^{bt}}{a-b}.$$

If $a = b$, then

$$\begin{aligned}(e^{at} * e^{at})(t) &= \int_0^t e^{at}\, e^{a(t-\tau)}\,d\tau \\ &= e^{at} \int_0^t 1\,d\tau \\ &= te^{at}\end{aligned}$$

□

These examples show that convolution is often easier to compute directly than it first appears, especially when the functions have simple forms.

7.2 The Laplace Transform and Convolution

Theorem 7.3. *Let f and g be functions for which the Laplace transforms exist. Define their convolution by*

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau.$$

Then the Laplace transform of the convolution is

$$\boxed{\mathscr{L}\{(f * g)(t)\}(s) = F(s)G(s),} \tag{16}$$

where

$$F(s) = \mathscr{L}\{f(t)\}(s), \qquad G(s) = \mathscr{L}\{g(t)\}(s).$$

Remark. Note that taking the inverse Laplace transform of both sides gives an equally useful formula:

$$\boxed{(f * g)(t) = \mathscr{L}^{-1}(F(s)G(s))} \tag{17}$$

Proof. Starting from the definition of the Laplace transform,

$$\mathscr{L}\{(f * g)(t)\}(s) = \int_0^\infty e^{-st} \left(\int_0^t f(\tau)\, g(t-\tau)\, d\tau \right) dt.$$

Using Fubini's theorem, we can swap the order of integration to find

$$\mathscr{L}\{(f * g)(t)\}(s) = \int_0^\infty \int_0^t e^{-st} f(\tau)\, g(t-\tau)\, d\tau\, dt.$$

The region of integration is

$$0 \le \tau \le t < \infty.$$

Rewriting this region by switching the order of integration gives

$$0 \le \tau < \infty, \qquad \tau \le t < \infty,$$

so

$$\mathscr{L}\{(f * g)(t)\}(s) = \int_0^\infty f(\tau) \left(\int_\tau^\infty e^{-st} g(t-\tau)\, dt \right) d\tau.$$

Now we make the change of variables

$$u = t - \tau, \qquad t = u + \tau.$$

This substitution is natural because the integrand contains $g(t-\tau)$, so replacing $t-\tau$ by a new variable simplifies the inner integral. When $t = \tau$, we get $u = 0$, and as $t \to \infty$, we have $u \to \infty$. Also,

$$dt = du.$$

If we view this as a two-variable change of variables,

$$(\tau, u) \mapsto (\tau, t) = (\tau, \tau + u),$$

then the Jacobian matrix is

$$\frac{\partial(t,\tau)}{\partial(u,\tau)} = \begin{pmatrix} \dfrac{\partial t}{\partial u} & \dfrac{\partial t}{\partial \tau} \\ \dfrac{\partial \tau}{\partial u} & \dfrac{\partial \tau}{\partial \tau} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Its determinant is

$$\left| \frac{\partial(t,\tau)}{\partial(u,\tau)} \right| = 1.$$

So, the area element does not pick up any extra factor:

$$dt\, d\tau = du\, d\tau.$$

Substituting $t = u + \tau$ into the integral gives

$$\mathscr{L}\{(f * g)(t)\}(s) = \int_0^\infty f(\tau) \left(\int_0^\infty e^{-s(u+\tau)} g(u)\, du \right) d\tau.$$

Since

$$e^{-s(u+\tau)} = e^{-s\tau} e^{-su},$$

we can separate the variables:

$$\mathscr{L}\{(f * g)(t)\}(s) = \left(\int_0^\infty f(\tau) e^{-s\tau}\, d\tau \right) \left(\int_0^\infty g(u) e^{-su}\, du \right).$$

Recognizing these as Laplace transforms, we obtain

$$\mathscr{L}\{(f * g)(t)\}(s) = F(s)G(s).$$

This proves the convolution theorem. □

7.3 A Physical Interpretation of Convolution

Convolution appears whenever a quantity built up over time is the result of many small contributions, where each contribution depends not only on *how much* was added, but also on *how long ago* it was added. In that setting, the present state is obtained by adding up all past inputs, each weighted by a response function that depends on the time lag.

More generally, suppose $u(t)$ is an input rate and $h(t)$ is a response kernel. Then a small amount $u(\tau)\, d\tau$ added at time τ contributes at later time t by the factor $h(t - \tau)$. Adding all such contributions gives

$$y(t) = \int_0^t u(\tau)\, h(t - \tau)\, d\tau,$$

which is exactly the convolution

$$y(t) = (u * h)(t).$$

Example 1: A decaying radioactive sample

Suppose a radioactive substance is continuously added at rate $u(t)$, and each bit of material decays exponentially after it is added. If a unit deposited at time τ survives until time t by the factor

$$h(t-\tau) = e^{-\lambda(t-\tau)},$$

then the amount remaining at time t from the material added near time τ is approximately

$$u(\tau)\,\Delta\tau\, e^{-\lambda(t-\tau)}.$$

Now divide the interval $[0, t]$ into n small pieces of width $\Delta\tau$, with sample points τ_i. The total amount remaining at time t is approximated by the discrete sum

$$y_n(t) = \sum_{i=0}^{n-1} u(\tau_i)\, e^{-\lambda(t-\tau_i)}\,\Delta\tau.$$

As the partition becomes finer, $\Delta\tau \to 0$, this Riemann sum converges to

$$y(t) = \int_0^t u(\tau)\, e^{-\lambda(t-\tau)}\, d\tau.$$

Thus the amount present at time t is the convolution of the input rate u with the decay kernel $e^{-\lambda t}$:

$$y(t) = (u * h)(t), \qquad h(t) = e^{-\lambda t}.$$

If there is also an initial amount y_0 present at time $t = 0$, then that initial mass contributes $y_0 e^{-\lambda t}$, so the full formula becomes

$$y(t) = y_0 e^{-\lambda t} + \int_0^t u(\tau) e^{-\lambda(t-\tau)}\, d\tau.$$

Example 2: Chick growth and feeding

Now suppose the chicks grow at a *linear rate* in response to feed. In other words, if a small amount of feed is given at time τ, then its contribution to the chicks' weight at a later time t is proportional to the amount of time that has passed since feeding. A simple model for this is

$$g(t-\tau) = \alpha(t-\tau),$$

where $\alpha > 0$ is a constant growth factor.

If $f(t)$ is the feed rate, then a small amount of feed given near time τ contributes approximately

$$f(\tau)\, g(t-\tau)\,\Delta\tau = f(\tau)\,\alpha(t-\tau)\,\Delta\tau$$

to the total growth at time t. Summing over many small intervals gives the discrete approximation

$$W_n(t) = \sum_{i=0}^{n-1} f(\tau_i)\,\alpha(t-\tau_i)\,\Delta\tau.$$

As the partition is refined and $\Delta\tau \to 0$, this Riemann sum converges to

$$W(t) = \int_0^t f(\tau)\, \alpha(t-\tau)\, d\tau.$$

Thus the growth is given by the convolution

$$W(t) = (f * g)(t), \qquad \text{with } g(s) = \alpha s.$$

This model says that earlier feed continues to contribute, but its effect changes linearly with the time elapsed since feeding. In this way, the total growth at time t is the accumulation of all past feeding contributions, each weighted by the linear growth response.

7.4 Using the convolution to solve differential equations

One of the most important applications of convolution is in solving linear differential equations, especially when the input (or forcing term) is not simple. The key idea is that convolution naturally encodes how a system responds over time to an external input.

Example 7.4. Solve the initial value problem

$$y'(t) + ay(t) = f(t), \qquad y(0) = y_0,$$

where $a > 0$ and $f(t)$ is a given forcing function.

Solution. We take the Laplace transform of both sides using formula (8) to find

$$sY(s) - y_0 + aY(s) = F(s),$$

so

$$Y(s) = \frac{F(s)}{s+a} + \frac{y_0}{s+a}.$$

Now, observe that

$$\mathscr{L}^{-1}\left\{\frac{1}{s+a}\right\} = e^{-at}.$$

It follows that

$$Y(s) = F(s) \cdot \frac{1}{s+a} + y_0 \cdot \frac{1}{s+a}.$$

We can utilize the convolution theorem (17) to find

$$\mathcal{L}^{-1}\{F(s) \cdot \tfrac{1}{s+a}\} = \int_0^t f(\tau)e^{-a(t-\tau)}\, d\tau,$$

and we obtain the solution

$$y(t) = y_0 e^{-at} + \int_0^t f(\tau)e^{-a(t-\tau)}\, d\tau.$$

This formula has a clear physical meaning. The term $y_0 e^{-at}$ represents the natural decay of the initial condition. The integral term represents the accumulated effect of the forcing function $f(t)$, where each past input $f(\tau)$ contributes to the present value $y(t)$, but is damped by the factor $e^{-a(t-\tau)}$. □

Example 7.5. Solve the initial value problem

$$y''(t) + y(t) = f(t), \qquad y(0) = 0, \qquad y'(0) = 0.$$

where $f(t)$ is an integrable function.

Solution. As usual, we take the Laplace transform of both sides using formula (9) to find

$$\mathscr{L}\{y''(t)\} + \mathscr{L}\{y(t)\} = \mathscr{L}\{f(t)\}.$$

Since $y(0) = 0$ and $y'(0) = 0$, we have

$$\mathscr{L}\{y''(t)\} = s^2 Y(s),$$

so the equation becomes

$$s^2 Y(s) + Y(s) = F(s).$$

Note that because the right hand side of the equation is given by some arbitrary function, we just write $F(s)$ to indicate its Laplace transform (we can assume that f has a Laplace transform if it is integrable). Factoring out $Y(s)$, we get

$$Y(s) = \frac{F(s)}{s^2 + 1}.$$

Now note that

$$\mathscr{L}^{-1}\left(\frac{1}{s^2 + 1}\right) = \sin(t).$$

It follows that we can write $Y(s)$ as

$$Y(s) = F(s) \cdot \mathscr{L}\{\sin t\}(s).$$

We can now utilize the convolution theorem (17)

$$y(t) = \int_0^t f(\tau) \sin(t - \tau)\, d\tau.$$

So, the solution is

$$y(t) = (f * \sin)(t) = \int_0^t f(\tau) \sin(t - \tau)\, d\tau.$$

The function $\sin t$ is the impulse response of the differential operator $D^2 + 1$. The solution at time t is obtained by adding up all past values of the forcing function $f(\tau)$, each weighted by the response $\sin(t - \tau)$. For example, if

$$f(t) = 1,$$

then

$$y(t) = \int_0^t \sin(t - \tau)\, d\tau.$$

Let $u = t - \tau$. Then $du = -d\tau$, and the integral becomes

$$y(t) = \int_0^t \sin u \, du = 1 - \cos t.$$

So the solution of

$$y'' + y = 1, \qquad y(0) = 0, \qquad y'(0) = 0$$

is

$$y(t) = 1 - \cos t.$$

□

This example shows how convolution turns the solution of a second-order differential equation into an integral against the system's response kernel. More generally, for a linear differential equation

$$L[y] = f(t),$$

the solution can often be written in the form

$$y(t) = \int_0^t f(\tau)h(t - \tau) \, d\tau,$$

where $h(t)$ is the *impulse response* of the system (the solution corresponding to a unit impulse input). In Laplace transform terms,

$$H(s) = \mathscr{L}\{h(t)\}(s) = \frac{1}{L(s)},$$

so that

$$Y(s) = F(s)H(s),$$

and hence

$$y(t) = (f * h)(t).$$