# Systems and Vector Spaces Practice Questions

1. Solve the following systems of equations

   (a)

$$\begin{aligned} 2x_1 + 5x_2 + 12x_3 &= 6 \\ 3x_1 + x_2 + 5x_3 &= 12 \\ 5x_1 + 8x_2 + 21x_3 &= 17 \end{aligned}$$

(b)

$$\begin{aligned} 3x_1 - 6x_2 - 2x_3 &= 1 \\ 2x_1 - 4x_2 + x_3 &= 17 \\ x_1 - 2x_2 - 2x_3 &= -9 \end{aligned}$$

(c)

$$\begin{aligned} 2x_1 + 8x_2 + 3x_3 &= 2 \\ x_1 + 3x_2 + 2x_3 &= 5 \\ 2x_1 + 7x_2 + 4x_3 &= 8 \end{aligned}$$

2. For the following systems, determine which values of $k$ yield (a) a unique solution, (b) no solution, and (c) infinitely many solutions.

   (a)

$$\begin{aligned} 3x + 2y &= 1 \\ 6x + 4y &= k \end{aligned}$$

(b)

$$\begin{aligned} x + 2y + z &= 3 \\ 2x - y - 3z &= 5 \\ 4x + 3y - z &= k \end{aligned}$$

3. For the following matrices $A$ and $B$, calculate $AB$ and $BA$.

(a) $A = \begin{pmatrix} 3 \\ -2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -2 \\ 3 & 1 \\ -4 & 5 \end{pmatrix}$

(b) $A = \begin{pmatrix} 2 & -1 \\ 3 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} -4 & 2 \\ 1 & 3 \end{pmatrix}$

4. Suppose $A$ and $B$ are square matrices such that $AB = BA$. Show that

$$(A+B)^2 = A^2 + 2AB + B^2$$

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Calculate $\sum_{n=0}^{\infty} A^n$.

6. Show that if $\mathbf{x_0}$ is solution to $\mathbf{Ax} = \mathbf{0}$ and $\mathbf{x_1}$ is a solution to $\mathbf{Ax} = \mathbf{b}$, then $\mathbf{x_0} + \mathbf{x_1}$ is also a solution to $\mathbf{Ax} = \mathbf{b}$.

7. Let $\mathbf{A}$ be an $n \times n$ matrix such that $\mathbf{Ax} = \mathbf{x}$ for every $n \times 1$ vector $\mathbf{x}$. Show that $\mathbf{A}$ is the identity matrix.

8. Consider the system $\mathbf{Ax} = \mathbf{b}$ where

$$\mathbf{A} = \begin{pmatrix} 3 & 7 \\ 2 & 5 \end{pmatrix} \qquad \mathbf{b} = \begin{pmatrix} -1 \\ 3. \end{pmatrix}$$

Find $\mathbf{A}^{-1}$ and use it to solve $\mathbf{Ax} = \mathbf{b}$.

9. Let $\mathbf{A}$ and $\mathbf{B}$ be invertible square matrices. Show the following identity

$$(\mathbf{A}^{-1} + \mathbf{B}^{-1})^{-1} = \mathbf{A}(\mathbf{A} + \mathbf{B})^{-1}\mathbf{B}$$

10. Calculate the following determinants.

(a) $\det\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$

(b) $\det\begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 5 & 0 \\ 3 & 6 & 9 & 8 \\ 4 & 0 & 10 & 7 \end{pmatrix}$

(c) $\det\begin{pmatrix} 2 & 0 & 0 & -3 \\ 0 & 1 & 11 & 12 \\ 0 & 0 & 5 & 7 \\ -4 & 0 & 0 & 7 \end{pmatrix}$

11. Write a formula that relates the determinants of the matrices **A** and **B**

$$\mathbf{A} = \begin{pmatrix} 1 & 7 & 6 \\ 2 & 3 & 4 \\ 1 & 1 & 1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 2 & 3 & 4 \\ 3 & 3 & 3 \\ 1 & 7 & 6. \end{pmatrix}$$

12. Suppose that $\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = 5$. Calculate the following determinants:

$$\det \begin{pmatrix} 2a & 2b & 2c \\ 3d - a & 3e - b & 3f - c \\ 4g + 3a & 4h + 3b & 4i + 3c \end{pmatrix} \qquad \det \begin{pmatrix} a + 2d & b + 2e & c + 2f \\ g & h & i \\ d & e & f \end{pmatrix}$$

13. Determine if the following vectors are linear independent over the corresponding vector spaces:

    (a) $\mathbf{u} = (0, 2), \mathbf{v} = (3, 0)$ over the vector space $\mathbb{R}^2$

    (b) $\mathbf{u} = (3, -1, 2), \mathbf{v} = (5, 4, -6), \mathbf{w} = (8, 3, -4)$ over the vector space $\mathbb{R}^3$.

    (c) $\mathbf{u} = (5, -2, 4), \mathbf{v} = (2, -3, 5), \mathbf{w} = (4, 5, -7)$ over the vector space $\mathbb{R}^3$.

    (d) $\mathbf{u} = (1, 1, 0), \mathbf{v} = (4, 3, 2), \mathbf{w} = (3, -2, -4)$ over the vector space $\mathbb{R}^3$.

    (e) $\mathbf{u} = (2, 0, 3), \mathbf{v} = (5, 4, -2), \mathbf{w} = (2, -1, 1)$ over the vector space $\mathbb{R}^3$.

14. Express the vector $\mathbf{t}$ as a linear combination of the vectors $\mathbf{v}$, $\mathbf{u}$, $\mathbf{w}$.

(a) $\mathbf{t} = (2, -7, 9), \mathbf{u} = (1, -2, 2), \mathbf{v} = (3, 0, 1), \mathbf{w} = (1, -1, 2)$ .

(b) $\mathbf{t} = (7, 7, 7), \mathbf{u} = (2, 5, 3), \mathbf{v} = (4, 1, -1), \mathbf{w} = (1, 1, 5)$.

15. Determine if the following spaces are vector subspaces.

   (a) $H = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0\}$

   (b) $H = \{(x, y, z) \in \mathbb{R}^3 \mid y = 1\}$

   (c) $H = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z = 1\}$

(d) $H = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$

(e) $H = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1 + x_2 = x_3 + x_4\}$

(f) $H = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1x_2 - x_3x_4 = 0\}$

16. Let $U$ and $V$ be subspaces of $\mathbb{R}^3$. We define the **intersection** $U \cap V$ as the space of all vectors that live in *both* $U$ and $V$. Show that $U \cap V$ is a vector subspace of $\mathbb{R}^3$.

17. Let $U$ and $V$ be subspaces of a vector space $W$. We define the **sum** $U + V$ as

$$U + V = \{u + v \mid u \in U, v \in V\}.$$

Show that $U + V$ is a vector subspace of $W$.