Higher Order Differential Equations

Contents

1 Introduction

Ordinary differential equations (ODEs) describe relationships between a function and its derivatives. Higher-order ODEs, involving derivatives of order two or higher, arise naturally in many applications such as mechanics, electrical circuits, and structural analysis.

An n-th order ODE has the general form:

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0. \tag{1}$$

The order of the equation is the highest derivative present. A higher-order ODE is **linear** if it can be written as:

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x). \tag{2}$$

Otherwise, it is **nonlinear**. Throughout this class, we will mainly concern ourselves with solving linear differential equations. Examples of nonlinear terms include $(y')^2$, yy'', or $\sin(y)$. If $g(x) = 0$, the equation is called **homogeneous**:

$$a_n(x)y^{(n)} + \cdots + a_0(x)y = 0. \tag{3}$$

If $g(x) \neq 0$, it is **nonhomogeneous**. We will first study how to solve homogeneous linear equations, and after study the inhomogenous case.

2 Complex Numbers and Functions

Complex numbers extend the real numbers and provide a natural framework for working with oscillatory functions such as $\sin x$ and $\cos x$. Complex numbers were first studied to understand the solutions to equations like

$$x^2 + 1 = 0.$$

It was obvious that a new type of number had to be used in order to solve this equation. The *fundamental* observation about adding the number i is that it fixes the problem for *every* equation. That is, including the number i allows for *every* polynomial with real coefficients to be solved!

Definition 2.1. A **complex number** is written as

$$z = a + bi, \quad a, b \in \mathbb{R}, \quad i^2 = -1.$$

We define the real and imaginary parts by $\Re(z) = a$ and $\Im(z) = b$.

Example 2.2. Consider the complex number $z = 2 + 3i$. It follows that

$$\Re(z) = 2 \qquad \Im(z) = 3.$$

2.1 Basic operations of complex numbers

Arithmetic with complex numbers is defined so that the usual algebraic rules hold, together with the identity $i^2 = -1$. We describe the basic operations in the following categories.
Addition and Subtraction. Complex numbers are added component-wise as follows:

$$(a+bi)+(c+di)=(a+c)+(b+d)i,$$

$$(a+bi)-(c+di)=(a-c)+(b-d)i.$$

Example 2.3.

$$(3+2i)+(1-5i)=(3+1)+i(2-5)=4-3i,$$
$$(3+2i)-(1-5i)=(3-1)+i(2-(-5))=2+7i.$$

Multiplication. Multiplication follows the distributive law as follows:

$$(a+bi)(c+di)=ac+adi+bci+bdi^2.$$

This follows the same pattern as if i were replaced by the variable x. The above expression can be simplified by using the identity that $i^2=-1$, which simplifies to

$$(a+bi)(c+di)=(ac-bd)+(ad+bc)i.$$

Example 2.4.

$$(2+3i)(1-4i)=2-8i+3i-12i^2=2-5i+12=14-5i.$$

The above calculation is something that is rather important in differential equations. It is good to practice this until you can do it rather quickly. It will also be important

Division. Division is defined by multiplying the numerator and denominator by the complex conjugate of the denominator.

Definition 2.5. Let $z = a+ib$ be a complex number. We define the **conjugate** of z, denoted as $\overline{z}$ as

$$\overline{z}=a-ib \tag{4}$$

If $z=a+bi$ and $w=c+di\neq 0$, then

$$\frac{z}{w}=\frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}=\frac{(a+bi)(c-di)}{c^2+d^2}.$$

Example 2.6.

$$\frac{2+3i}{1-2i}=\frac{(2+3i)(1+2i)}{1^2+(-2)^2}=\frac{2+4i+3i+6i^2}{5}=\frac{2+7i-6}{5}=\frac{-4+7i}{5}.$$

Proposition 2.7. *Let $z=a+ib$ be a complex number. Then, z and $\overline{z}$ satisfy the following properties:*

1. $z + \overline{z} = 2a$

2. $z\overline{z} = a^2 + b^2 = |z|^2$

Example 2.8.

$$(3 + 4i)(3 - 4i) = 9 + 16 = 25.$$

Definition 2.9. The **modulus** (sometimes called the magnitude) of a complex number $z = a + bi$ is defined by

$$|z| \coloneqq \sqrt{a^2 + b^2}. \tag{5}$$

Geometrically, this is the distance from the origin to the point (a, b) in the plane.

Complex numbers can therefore be viewed as points or vectors in $\mathbb{R}^2$, where:

- addition corresponds to vector addition,
- multiplication combines scaling and rotation.

2.2 The Complex Exponential

The complex exponential function is one of the most important tools in applied mathematics. It provides a unified way to describe both exponential growth/decay and oscillatory behavior (sine and cosine). This makes it especially useful in many areas of mathematics, physics, and engineering. First, we recall the definition of the exponential function from its power series definition.

Definition 2.10. For $z \in \mathbb{C}$, the **complex exponential** is defined by the power series

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{6}$$

This definition is identical to the real exponential, but now z is allowed to be complex. A key fact (which we will not prove here) is that this series converges for every complex number z. Thus, e^z is defined for all $z \in \mathbb{C}$. One advantage of this definition is that it immediately implies many familiar properties, such as

$$e^{z+w} = e^z e^w, \quad e^0 = 1.$$

Let $z = x + iy$, where $x, y \in \mathbb{R}$. Using properties of exponents, we write

$$e^z = e^{x+iy} = e^x e^{iy}.$$

Thus, understanding e^{iy} is the key to understanding the complex exponential.

Definition 2.11. For $\theta \in \mathbb{R}$, Euler's formula states

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{7}$$

We now justify Euler's formula using power series. Recall the power series expansions for the exponential, cosine, and sine functions:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \cos\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n}}{(2n)!}, \quad \sin\theta = \sum_{n=0}^{\infty} \frac{(-1)^n \theta^{2n+1}}{(2n+1)!}.$$

Now substitute $z = i\theta$ into the exponential series:

$$e^{i\theta} = \sum_{n=0}^{\infty} \frac{(i\theta)^n}{n!}.$$

We separate this into even and odd powers of n.

Even powers ($n = 2k$):

$$(i\theta)^{2k} = i^{2k}\theta^{2k} = (-1)^k \theta^{2k}.$$

Thus, the even terms become

$$\sum_{k=0}^{\infty} \frac{(i\theta)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k}}{(2k)!} = \cos\theta.$$

Odd powers ($n = 2k + 1$):

$$(i\theta)^{2k+1} = i^{2k+1}\theta^{2k+1} = i(-1)^k \theta^{2k+1}.$$

Thus, the odd terms become

$$\sum_{k=0}^{\infty} \frac{(i\theta)^{2k+1}}{(2k+1)!} = i\sum_{k=0}^{\infty} \frac{(-1)^k \theta^{2k+1}}{(2k+1)!} = i\sin\theta.$$

Adding the even and odd parts together, we obtain

$$e^{i\theta} = \cos\theta + i\sin\theta.$$

This derivation shows that Euler's formula is not mysterious—it follows directly from the power series definitions of the exponential, sine, and cosine functions.

This remarkable identity connects exponential functions with trigonometric functions. It can be derived by substituting $z = i\theta$ into the power series for e^z and comparing with the power series for $\cos\theta$ and $\sin\theta$.

Combining this with the previous subsection gives the fundamental identity:

$$e^{x+iy} = e^x(\cos y + i\sin y).$$

This formula is extremely important: it shows that the real part x controls exponential growth or decay, while the imaginary part y controls oscillation.

Example 2.12. To compute the complex exponential e^{2+3i}, we use Euler's formula to find

$$e^{2+3i} = e^2(\cos 3 + i\sin 3).$$

This expresses the complex number in terms of a magnitude e^2 and an angle 3.

Euler's formula can be rearranged to express sine and cosine in terms of exponentials:

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}, \quad \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}.$$

These identities are extremely useful because exponentials are often easier to manipulate than trigonometric functions.

Example 2.13. Prove the formula

$$\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$$

Proof. Now write

$$e^{i(x+y)} = e^{ix}e^{iy}.$$

Applying Euler's formula to each factor gives

$$\cos(x+y) + i\sin(x+y) = (\cos x + i\sin x)(\cos y + i\sin y).$$

Multiplying the right-hand side,

$$(\cos x + i\sin x)(\cos y + i\sin y) = (\cos x\cos y - \sin x\sin y) + i(\sin x\cos y + \cos x\sin y).$$

Since two complex numbers are equal if and only if their real and imaginary parts are equal, we compare imaginary parts and obtain

$$\sin(x+y) = \sin x\cos y + \cos x\sin y.$$

This is the desired identity. □

The complex exponential also allows for the evaluation of integrals.

Example 2.14. Evaluate the integral

$$\int e^x\cos(x)\;dx$$

Recall that

$$\cos x = \Re(e^{ix}),$$

so

$$e^x\cos x = \Re\big(e^xe^{ix}\big) = \Re\big(e^{(1+i)x}\big).$$

Thus,

$$\int e^x\cos x\,dx = \Re\left(\int e^{(1+i)x}\,dx\right).$$

Now compute the integral:

$$\int e^{(1+i)x}dx = \frac{1}{1+i}e^{(1+i)x}.$$

Simplify the coefficient:

$$\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2}.$$

So,

$$\int e^{(1+i)x}dx = \frac{1-i}{2}e^x(\cos x + i\sin x).$$

Now take the real part:

$$\frac{1}{2}e^x\big[(\cos x + \sin x) + i(\sin x - \cos x)\big].$$

Therefore,

$$\int e^x \cos x\,dx = \frac{e^x}{2}(\cos x + \sin x) + C.$$

The above method is known as **complexifying the integral**.

2.3 Geometric Interpretation

The formula

$$e^{i\theta} = \cos\theta + i\sin\theta$$

describes a point on the unit circle in the complex plane. As θ varies, $e^{i\theta}$ traces out the unit circle.

Thus:

- $|e^{i\theta}| = 1$ for all θ,
- multiplying by $e^{i\theta}$ corresponds to a rotation by angle θ.

More generally,

$$e^{x+iy} = e^x e^{iy}$$

represents a number with magnitude e^x and angle y.

Example 2.15. Multiplying a complex number by $e^{i\pi/2} = i$ rotates it by 90° counterclockwise.

Definition 2.16. The complex exponential is periodic in the imaginary direction with period $2\pi i$:

$$e^{z+2\pi i} = e^z.$$

This follows from the periodicity of sine and cosine:

$$\cos(\theta + 2\pi) = \cos\theta, \quad \sin(\theta + 2\pi) = \sin\theta.$$

Example 2.17.

$$e^{i(\theta+2\pi)} = e^{i\theta}.$$

If $z = x + iy$, then

$$|e^z| = |e^x e^{iy}| = e^x.$$

This shows:

- The real part x determines the size (growth or decay),
- The imaginary part y affects only the angle (oscillation).

Example 2.18.

$$|e^{-2+5i}| = e^{-2}.$$

Even though the number oscillates (because of $5i$), its magnitude is determined entirely by -2.

3 The Wronskian and Linear Independence

When solving linear differential equations, it is essential to determine whether a collection of functions is **linearly independent**. This is because the general solution is built from a set of independent solutions, and redundancy must be avoided.The main tool for testing linear independence of functions is the **Wronskian**.

Definition 3.1. Let y_1, y_2 be differentiable functions. The **Wronskian** of y_1 and y_2 is defined by

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)y_2'(x) - y_1'(x)y_2(x).$$

More generally, for n functions $y_1, \ldots, y_n$, the Wronskian is the determinant

$$W(y_1, \ldots, y_n)(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

The Wronskian provides a test for linear independence:

Theorem 3.2. *Let y_1 and y_2 be two differentiable functions. We have the following criteron for determining linear independence.*

- *If $W(y_1, y_2)(x_0) \neq 0$ for some x_0, then y_1 and y_2 are linearly independent.*
- *If y_1 and y_2 are linearly dependent, then $W(y_1, y_2)(x) \equiv 0$ for all x.*

Caution: The Wronskian Does Not Fully Characterize Linear Dependence

While the Wronskian is a useful tool for detecting linear independence, it is important to understand its limitations.

- If $W(y_1, y_2)(x_0) \neq 0$ for some x_0, then y_1 and y_2 are linearly independent.
- However, the converse is *not always true*: it is possible for $W(y_1, y_2)(x) \equiv 0$ even though y_1 and y_2 are linearly independent.

This means that a zero Wronskian does *not* necessarily imply linear dependence unless additional assumptions are satisfied.

Example 3.3. Define

$$y_1(x) = x^2, \qquad y_2(x) = \begin{cases} x^2, & x \geq 0, \\ -x^2, & x < 0. \end{cases}$$

Both functions are differentiable everywhere, and one can compute that

$$W(y_1, y_2)(x) = 0 \quad \text{for all } x.$$

However, y_2 is not a constant multiple of y_1 on all of $\mathbb{R}$, so the functions are linearly independent.

Thus, for solutions of linear differential equations, the Wronskian *does* correctly determine linear dependence. Thus, a nonzero Wronskian guarantees independence. To understand why the Wronskian works, recall that two functions are linearly dependent if one is a scalar multiple of the other:

$$y_2(x) = c\, y_1(x).$$

Differentiating gives

$$y_2'(x) = c\, y_1'(x).$$

Thus, the second column of the Wronskian matrix is a multiple of the first:

$$\begin{pmatrix} y_2 \\ y_2' \end{pmatrix} = c \begin{pmatrix} y_1 \\ y_1' \end{pmatrix}.$$

This means the columns are linearly dependent vectors, so the determinant is zero:

$$W(y_1, y_2) = 0.$$

Conversely, if the determinant is nonzero, then the columns are linearly independent as vectors in $\mathbb{R}^2$, which implies the functions themselves are linearly independent. The Wronskian can be interpreted geometrically. For two functions, the determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

represents the signed area of the parallelogram formed by the vectors

$$(y_1, y_1'), \quad (y_2, y_2').$$

- If the area is zero, the vectors lie on the same line ⇒ the functions are dependent.
- If the area is nonzero, the vectors span the plane ⇒ the functions are independent.

Example 3.4. Determine whether $\cos x$ and $\sin x$ are linearly independent.

Compute the Wronskian:

$$W(\cos x, \sin x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1.$$

4 Solving Constant-Coefficient Homogeneous ODEs

A **constant-coefficient homogeneous linear differential equation** is an equation of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = 0, \tag{8}$$

where $a_0, a_1, \ldots, a_n$ are constants and $a_n \neq 0$. These equations are among the most important in the theory of ordinary differential equations because they arise naturally in applications and can often be solved explicitly.

The key feature of such equations is that their coefficients do not depend on the independent variable. This allows us to look for solutions of exponential form. People often wonder why this is the function to guess. Let's consider a second order ODE:

$$y'' + ay' + by = 0$$

Examining this ODE, we want to find a function such that its derivatives differ only by a constant. If this were not the case, that is, if y' had no relation to y, how could we ever expect to sum y' and y to zero? The only function whose derivative is constant multiple of itself is the exponential.

We now try to solve equation 8 by substituting $y = e^{rx}$. This yields

$$a_n r^n e^{rx} + a_{n-1} r^{n-1} e^{rx} + \cdots + a_1 r e^{rx} + a_0 e^{rx} = 0. \tag{9}$$

As the exponential is never equal to zero, equation 9 boils down to solving the following polynomial

$$a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 \tag{10}$$

Definition 4.1. Given an constant coefficient differential equation given by equation 8, we call the polynomial

$$p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0 \tag{11}$$

as the **characteristic polynomial**.

Solving the differential equation is therefore reduced to solving an algebraic polynomial equation.

We can now discuss the details of the solution to equation 8 from the behavior of its characteristic polynomial. If the characteristic equation has n roots, counted with multiplicity, then the differential equation has n linearly independent solutions. These solutions form a basis for the solution space, and the general solution is a linear combination of them:

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x), \tag{12}$$

where $C_1, C_2, \ldots, C_n$ are arbitrary constants. This fact is known as the **superposition principle** and is *crucial* for solving linear differential equations.

There are three different situations that can arise when finding the roots of polynomials: the roots are all distinct, there are multiple of the same root, and there exists complex roots. Each one of these situations produces rather different solutions. We now discuss each of these in the following sections.

4.1 Distinct Real Roots

We begin with the case in which the roots of the characteristic polynomial are all real and distinct. Suppose that the characteristic equation has distinct real roots

$$r_1, r_2, \ldots, r_n.$$

Then the corresponding linearly independent solutions are

$$e^{r_1 x},\ e^{r_2 x},\ \ldots,\ e^{r_n x}.$$

Hence, the general solution is

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \cdots + C_n e^{r_n x}.} \tag{13}$$

We now check that each of the solutions is linearly independent. This is done by computing its Wronskian. We present the result in the following theorem.

Theorem 4.2. *Let $r_1, \ldots, r_n$ be distinct real numbers. Then the functions*

$$f_1(x) = e^{r_1 x}, \quad f_2(x) = e^{r_2 x}, \quad \ldots, \quad f_n(x) = e^{r_n x}$$

are linearly independent.

Proof. Consider the Wronskian of these functions:

$$W(f_1, \ldots, f_n)(x) = \det \begin{pmatrix} e^{r_1 x} & e^{r_2 x} & \cdots & e^{r_n x} \\ r_1 e^{r_1 x} & r_2 e^{r_2 x} & \cdots & r_n e^{r_n x} \\ r_1^2 e^{r_1 x} & r_2^2 e^{r_2 x} & \cdots & r_n^2 e^{r_n x} \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} e^{r_1 x} & r_2^{n-1} e^{r_2 x} & \cdots & r_n^{n-1} e^{r_n x} \end{pmatrix}.$$

Factor $e^{r_j x}$ out of the j-th column. Then

$$W(f_1, \ldots, f_n)(x) = (e^{r_1 x} e^{r_2 x} \cdots e^{r_n x}) \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{pmatrix}.$$

Since $e^{r_1 x} \cdots e^{r_n x} = e^{(r_1 + \cdots + r_n)x}$, we get

$$W(f_1, \ldots, f_n)(x) = e^{(r_1 + \cdots + r_n)x} \det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ r_1 & r_2 & \cdots & r_n \\ r_1^2 & r_2^2 & \cdots & r_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_n^{n-1} \end{pmatrix}.$$

The determinant on the right is the Vandermonde determinant, which equals

$$\prod_{1\le i<j\le n} (r_j - r_i).$$

Because the r_i are distinct, every factor $r_j - r_i \neq 0$, so

$$\prod_{1\le i<j\le n} (r_j - r_i) \neq 0.$$

Also, $e^{(r_1+\cdots+r_n)x} \neq 0$ for all x. Therefore,

$$W(f_1, \ldots, f_n)(x) \neq 0 \quad \text{for all } x.$$

Hence the functions $e^{r_1 x}, \ldots, e^{r_n x}$ are linearly independent. □

Now that we have shown that the solutions are indeed linearly independent, we present some examples.

Example 4.3. Consider the differential equation

$$y'' - 5y' + 6y = 0.$$

The characteristic equation is

$$r^2 - 5r + 6 = 0,$$

which factors as

$$(r-2)(r-3) = 0.$$

Thus, the roots are $r_1 = 2$ and $r_2 = 3$, which are distinct. The general solution is

$$y(x) = C_1 e^{2x} + C_2 e^{3x}.$$

Example 4.4. Consider the differential equation

$$y''' - 6y'' + 11y' - 6y = 0.$$

The characteristic equation is

$$r^3 - 6r^2 + 11r - 6 = 0.$$

Factoring gives

$$(r-1)(r-2)(r-3) = 0.$$

Thus, the roots are $r_1 = 1$, $r_2 = 2$, and $r_3 = 3$, all distinct. The general solution is

$$y(x) = C_1 e^{x} + C_2 e^{2x} + C_3 e^{3x}.$$

4.2 Repeated Real Roots and Reduction of Order

A more subtle case occurs when the characteristic equation has a repeated root. Suppose r is a root of multiplicity m. Then the associated solutions are

$$e^{rx},\ xe^{rx},\ x^2e^{rx},\ \ldots,\ x^{m-1}e^{rx}. \tag{14}$$

Thus, a repeated root of multiplicity m contributes m linearly independent solutions. To understand why this is, we consider the case of a repeated root for a second order system. In the case of repeated roots of the characteristic equation, the method of **reduction of order** provides a systematic way to construct additional linearly independent solutions. Suppose that we have a second-order linear homogeneous equation

$$y'' + a_1(x)y' + a_0(x)y = 0,$$

and suppose that one nontrivial solution $y_1(x)$ is already known. We seek a second solution of the form

$$y_2(x) = v(x)\, y_1(x),$$

where $v(x)$ is an unknown function to be determined. Let

$$y_2 = vy_1.$$

It follows that

$$y_2' = v'y_1 + vy_1',$$

$$y_2'' = v''y_1 + 2v'y_1' + vy_1''.$$

Substituting into the differential equation gives

$$v''y_1 + 2v'y_1' + vy_1'' + a_1(x)(v'y_1 + vy_1') + a_0(x)vy_1 = 0.$$

Grouping terms gives,

$$v''y_1 + v'\big(2y_1' + a_1(x)y_1\big) + v\big(y_1'' + a_1(x)y_1' + a_0(x)y_1\big) = 0.$$

Since y_1 is a solution, it satisfies

$$y_1'' + a_1(x)y_1' + a_0(x)y_1 = 0,$$

so the last term vanishes. Therefore,

$$v''y_1 + v'\big(2y_1' + a_1(x)y_1\big) = 0.$$

Let $w = v'$. Then we obtain a first-order equation:

$$w'y_1 + w\big(2y_1' + a_1(x)y_1\big) = 0.$$

Dividing by y_1,

$$w' + w\left(2\frac{y_1'}{y_1} + a_1(x)\right) = 0.$$

This is a first-order linear equation for w, which can be solved by standard methods. Once w is found, we integrate to obtain v, and hence $y_2 = vy_1$. Now consider the constant-coefficient equation

$$y'' - 2ry' + r^2y = 0.$$

The characteristic equation is

$$(r-\lambda)^2 = 0,$$

so there is a repeated root $\lambda = r$. A known solution is

$$y_1(x) = e^{rx}.$$

We now use reduction of order to find a second solution. Assume

$$y_2 = v(x)e^{rx}.$$

We compute the derivatives of y_2 to find

$$y_2' = v'e^{rx} + rve^{rx},$$

$$y_2'' = v''e^{rx} + 2rv'e^{rx} + r^2ve^{rx}.$$

Substituting into the differential equation yields

$$y_2'' - 2ry_2' + r^2y_2 = 0$$

which gives

$$(v''e^{rx} + 2rv'e^{rx} + r^2ve^{rx}) - 2r(v'e^{rx} + rve^{rx}) + r^2ve^{rx} = 0.$$

Simplifying yields

$$v''e^{rx} + 2rv'e^{rx} + r^2ve^{rx} - 2rv'e^{rx} - 2r^2ve^{rx} + r^2ve^{rx} = 0.$$

and hence

$$v''e^{rx} = 0.$$

Since $e^{rx} \neq 0$, we obtain

$$v'' = 0.$$

It follow from taking a linearly independent solution, we have

$$y_2 = xe^{rx}.$$

We have shown that when a root r is repeated, the second solution takes the form

$$y_2 = xe^{rx}.$$

Thus, for a repeated root r, the general solution is

$$\boxed{y(x) = (C_1 + C_2x)e^{rx}.}$$

This method generalizes: for a root of multiplicity m, repeated application of reduction of order produces solutions

$$e^{rx}, \quad xe^{rx}, \quad x^2e^{rx}, \quad \ldots, \quad x^{m-1}e^{rx}.$$

Reduction of order therefore provides a theoretical justification for the appearance of polynomial factors in the repeated root case. We the have the following general rule: if r is a root of multiplicity m,then the solutions are $x^k e^{rx}, \quad k = 0, 1, \ldots, m-1$. We consider the following examples.

Example 4.5. Consider the second-order homogeneous differential equation

$$y'' - 4y' + 4y = 0.$$

The characteristic equation is

$$r^2 - 4r + 4 = 0,$$

which factors as

$$(r-2)^2 = 0.$$

Thus, we have a repeated root $r = 2$. For a repeated root r, the general solution has the form

$$y(t) = (C_1 + C_2t)e^{rt}.$$

Therefore, the general solution is

$$y(t) = (C_1 + C_2t)e^{2t}.$$

Example 4.6. Consider the third-order homogeneous differential equation

$$y''' - 3y'' + 3y' - y = 0.$$

The characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = 0,$$

which factors as

$$(r-1)^3 = 0.$$

Thus, we have a triple repeated root at $r = 1$. For a root r of multiplicity 3, the general solution has the form

$$y(t) = \left(C_1 + C_2t + C_3t^2\right)e^{rt}.$$

Therefore, the general solution is

$$y(t) = \left(C_1 + C_2t + C_3t^2\right)e^{t}.$$

4.3 Complex Roots

We have now considered the cases of distinct real roots and repeated real roots. The final possibility for the characteristic equation is that it has **complex (non-real) roots**. This completes our classification of all possible behaviors for linear differential equations with constant coefficients.

In this section, we study how complex roots arise and, more importantly, how they produce **real-valued, linearly independent solutions**. A central theme is that even though the roots and intermediate steps may involve complex numbers, the final solutions can always be written in real form.

4.3.1 Purely Imaginary Roots

We begin with the simplest complex case: purely imaginary roots. Suppose the characteristic equation has roots

$$r = \pm i\beta, \quad \beta > 0.$$

Then the corresponding solutions (formally) are

$$y_1(t) = e^{i\beta t}, \quad y_2(t) = e^{-i\beta t}.$$

At first glance, these are complex-valued functions. However, using Euler's formula,

$$e^{i\beta t} = \cos(\beta t) + i\sin(\beta t), \quad e^{-i\beta t} = \cos(\beta t) - i\sin(\beta t),$$

we see that each exponential contains both cosine and sine. The key observation is the following:

If a differential equation has real coefficients, then the real and imaginary parts of any complex solution are themselves solutions.

Applying this to $e^{i\beta t}$, we immediately obtain two real-valued solutions:

$$\cos(\beta t), \quad \sin(\beta t).$$

Thus, from the pair of complex exponentials $e^{\pm i\beta t}$, we extract two real functions. A second-order differential equation requires two **linearly independent** solutions to form the general solution. It is therefore essential that $\cos(\beta t)$ and $\sin(\beta t)$ are independent. This can be verified using the Wronskian:

$$W(\cos(\beta t), \sin(\beta t)) = \begin{vmatrix} \cos(\beta t) & \sin(\beta t) \\ -\beta\sin(\beta t) & \beta\cos(\beta t) \end{vmatrix} = \beta \neq 0.$$

Because the Wronskian is nonzero, the functions are linearly independent and form a fundamental set of solutions.

Remark. A single complex exponential $e^{i\beta t}$ already contains two real functions (its real and imaginary parts). This is why a *pair* of complex conjugate roots produces exactly the number of independent solutions that we need.

Another way to see this structure is to combine the two exponentials:

$$\cos(\beta t) = \frac{e^{i\beta t} + e^{-i\beta t}}{2}, \quad \sin(\beta t) = \frac{e^{i\beta t} - e^{-i\beta t}}{2i}.$$

These formulas show explicitly how cosine and sine arise from the pair of conjugate exponentials. The two exponentials are not independent over the real numbers; rather, they combine to produce two independent real-valued functions.

If the roots are $\pm i\beta$, then the general solution is

$$y(t) = C_1\cos(\beta t) + C_2\sin(\beta t).$$

This form is entirely real and is the standard way of writing solutions in this case.

Example 4.7. Consider the differential equation

$$y'' + 4y = 0.$$

The characteristic equation is

$$r^2 + 4 = 0,$$

so

$$r = \pm 2i.$$

The corresponding complex solutions are e^{2it} and e^{-2it}. Using Euler's formula, we obtain the real solutions

$$\cos(2t), \quad \sin(2t).$$

Thus, the general solution is

$$\boxed{y(t) = C_1 \cos(2t) + C_2 \sin(2t).}$$

4.3.2 General Complex Roots

We now consider the more general case where the roots are

$$r = \alpha \pm i\beta, \quad \beta > 0.$$

The corresponding complex solutions are

$$e^{(\alpha+i\beta)t}, \quad e^{(\alpha-i\beta)t}.$$

Using Euler's formula,

$$\begin{aligned} e^{(\alpha+i\beta)t} &= e^{\alpha t}(\cos(\beta t) + i\sin(\beta t)), \\ e^{(\alpha-i\beta)t} &= e^{\alpha t}(\cos(\beta t) - i\sin(\beta t)). \end{aligned}$$

Taking real and imaginary parts, we obtain two real solutions:

$$e^{\alpha t}\cos(\beta t), \quad e^{\alpha t}\sin(\beta t).$$

These two functions are again linearly independent. Intuitively:

- $e^{\alpha t}$ controls growth or decay,
- $\cos(\beta t)$ and $\sin(\beta t)$ provide oscillation,
- their combination produces two distinct behaviors that cannot be written as multiples of each other.

Thus, they form a fundamental set of solutions. If the roots are $\alpha \pm i\beta$, then the general solution is

$$\boxed{y(t) = e^{\alpha t}\big(C_1 \cos(\beta t) + C_2 \sin(\beta t)\big).}$$

Example 4.8. Consider the differential equation

$$y'' - 2y' + 5y = 0.$$

The characteristic equation is

$$r^2 - 2r + 5 = 0,$$

which gives

$$r = 1 \pm 2i.$$

Thus, the general solution is

$$y(t) = e^t \big(C_1 \cos(2t) + C_2 \sin(2t)\big).$$

This case completes our analysis of all possible roots. The key takeaways are:

- Complex roots always occur in conjugate pairs.
- Each pair produces **two linearly independent real solutions**.
- Complex exponentials naturally encode both exponential growth/decay and oscillation.

Thus, even though complex numbers appear in the intermediate steps, they ultimately provide a powerful and efficient way to construct real solutions.

If the characteristic equation has complex roots, they occur in conjugate pairs because the coefficients of the polynomial are real. Suppose

$$r = \alpha \pm i\beta, \tag{15}$$

where α and β are real and $\beta \neq 0$.

The corresponding exponential solutions are

$$e^{(\alpha+i\beta)x} \quad \text{and} \quad e^{(\alpha-i\beta)x}. \tag{16}$$

Using Euler's formula,

$$e^{i\beta x} = \cos(\beta x) + i\sin(\beta x), \tag{17}$$

we can rewrite these complex exponentials in terms of real-valued functions. The real linearly independent solutions are

$$e^{\alpha x}\cos(\beta x) \quad \text{and} \quad e^{\alpha x}\sin(\beta x). \tag{18}$$

Thus, the real general solution associated with the pair $\alpha \pm i\beta$ is

$$y(x) = e^{\alpha x}\left(C_1 \cos(\beta x) + C_2 \sin(\beta x)\right). \tag{19}$$

These solutions describe oscillation with exponential growth or decay depending on the sign of α. If $\alpha > 0$, the oscillations grow; if $\alpha < 0$, they decay.

4.4 Repeated Complex Roots

We have seen how distinct complex roots $\alpha \pm i\beta$ lead to solutions of the form

$$e^{\alpha t}\cos(\beta t), \quad e^{\alpha t}\sin(\beta t).$$

We now consider the case where complex roots occur with **multiplicity greater than one**. This is the natural extension of repeated real roots to the complex setting. Suppose the characteristic equation has a complex root

$$r = \alpha + i\beta$$

of multiplicity $s \geq 2$. Since the coefficients of the differential equation are real, the conjugate root

$$\alpha - i\beta$$

must also appear with the same multiplicity s. Thus, the characteristic polynomial contains the factor

$$\big((r-(\alpha+i\beta))(r-(\alpha-i\beta))\big)^s = \big((r-\alpha)^2+\beta^2\big)^s.$$

From the theory of repeated roots, each root of multiplicity s produces s independent solutions of the form

$$t^k e^{rt}, \quad k = 0, 1, \ldots, s-1.$$

Applying this to the complex root $r = \alpha + i\beta$, we obtain

$$e^{(\alpha+i\beta)t}, \quad te^{(\alpha+i\beta)t}, \quad \ldots, \quad t^{s-1}e^{(\alpha+i\beta)t}.$$

Similarly, from the conjugate root we obtain

$$e^{(\alpha-i\beta)t}, \quad te^{(\alpha-i\beta)t}, \quad \ldots, \quad t^{s-1}e^{(\alpha-i\beta)t}.$$

Together, these give $2s$ complex-valued solutions. As before, we convert these into real-valued solutions using Euler's formula:

$$e^{(\alpha\pm i\beta)t} = e^{\alpha t}(\cos(\beta t) \pm i\sin(\beta t)).$$

Multiplying by powers of t, we obtain expressions of the form

$$t^k e^{\alpha t}(\cos(\beta t) \pm i\sin(\beta t)).$$

Taking real and imaginary parts, we obtain the real solutions:

$$t^k e^{\alpha t}\cos(\beta t), \quad t^k e^{\alpha t}\sin(\beta t), \quad k = 0, 1, \ldots, s-1.$$

Thus, for a repeated complex root $\alpha \pm i\beta$ of multiplicity s, the general real-valued solution is

$$y(t) = \sum_{k=0}^{s-1} \left[C_{2k+1}\, t^k e^{\alpha t}\cos(\beta t) + C_{2k+2}\, t^k e^{\alpha t}\sin(\beta t)\right].$$

In particular:

- For $s = 1$, we recover the usual pair

$$e^{\alpha t}\cos(\beta t), \quad e^{\alpha t}\sin(\beta t).$$

- For $s = 2$, we obtain four solutions:

$$e^{\alpha t}\cos(\beta t), \quad e^{\alpha t}\sin(\beta t), \quad te^{\alpha t}\cos(\beta t), \quad te^{\alpha t}\sin(\beta t).$$

The key idea is that repeated roots require additional independent solutions. In the real case, we multiply by powers of t to obtain new solutions. The same idea applies here, but each complex exponential already contains two real functions (cosine and sine). The functions

$$t^k e^{\alpha t}\cos(\beta t), \quad t^k e^{\alpha t}\sin(\beta t)$$

are linearly independent for different values of k. Intuitively, each additional factor of t introduces qualitatively different behavior that cannot be replicated by a linear combination of lower-order terms.

More formally, one can show that the Wronskian of these functions is nonzero, so they form a fundamental set of solutions.

Example 4.9. Solve the differential equation with characteristic equation

$$(r^2 + 2r + 5)^2 = 0.$$

Solution. First, solve

$$r^2 + 2r + 5 = 0,$$

which gives

$$r = -1 \pm 2i.$$

Since the factor is squared, the root has multiplicity 2. Thus, the general solution is

$$y(t) = e^{-t}\big(C_1\cos(2t) + C_2\sin(2t)\big) + te^{-t}\big(C_3\cos(2t) + C_4\sin(2t)\big).$$

□

5 Theory of Existence and Uniqueness

We are going to consider the case of second order constant coefficient ODEs:

$$y'' + py' + qy = 0$$

First, we have seen that solutions to the above differential equation are of the form

$$y = c_1 y_1 + c_2 y_2$$

where y_1 and y_2 are *linearly independent*. The first natural question to ask is: *why are these all of the solutions?* The answer to this question is answered by the **superposition principle**. This is stated as follows.

Proposition 5.1. *If y_1 and y_2 are solutions to a linear homogeneous differential equation, then*

$$c_1y_1 + c_2y_2$$

is also a solution to the same differential equation.

This is the natural place to take a slight detour and discuss an object called a linear operator.

5.1 Linear Operators

To motivate this idea, we want to write the differential equation of interest in a slightly different manner. We have the differential equation in the form

$$y'' + py' + qy = 0.$$

We are going to make the following *cosmetic* changes in the following way. First, instead of writing y'' ad y', we write $y'' = \frac{d^2}{dt^2}y$ and $y' = \frac{d}{dt}y$ which yields

$$\frac{d^2}{dt^2}y + p\frac{d}{dt}y + qy = 0$$

Writing derivatives in this way can be a little cumbersome. So, we write $D^2 := \frac{d^2}{dt^2}$ and $D := \frac{d}{dt}$, which yields

$$D^2y + pDy + y = 0.$$

Writing the differential equation in this manner allows us to formally *factor* the variable y out of the equation as such

$$\left(D^2 + pD + q\right)y = 0$$

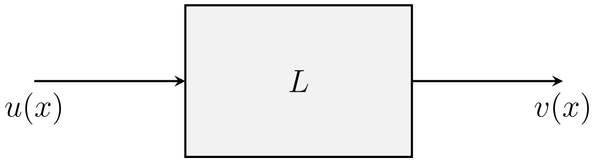
Everyone reads this as $D^2 + pD + q$ *times* y, as the notation suggests. However, what is happening is that we are *applying* D^2+pD+q to y. With this in mind, we define D^2+pD+q as an object all by itself, called a **linear operator**. Usually, this is written as

$$L = D^2 + pD + q$$

and hence we can write the differential equation as

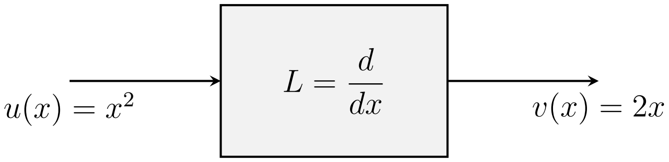
$$L[y] = 0.$$

The way to think of L is as follows: there is a function $u(x)$ that goes *into* the linear operator L and it spits out another function $v(x)$. This is shown in the diagram below.

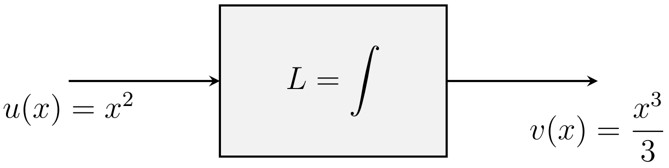


To get a better feel for this, consider the following examples of linear operators.

Example 5.2. The derivative is a linear operator. If we set $L = \frac{d}{dx}$ and consider $u(x) = x^2$, we have the following diagram:



Example 5.3. Another important example is viewing the integral as a linear operator. If we set $L = \int$ and and consider $u(x) = x^2$, we have the following diagram:



You might be reading this and wonder how did I determine that the derivative and the integral were linear operators. There are two rules that an operator must obey in order to be a linear operator.

Definition 5.4. An operator L is a **linear operator** if

1. for functions u_1 and u_2,
$$L(u_1 + u_2) = L(u_1) + L(u_2), \tag{20}$$
and
2. for a function u and constant $c \in \mathbb{R}$,
$$L(c \cdot u) = cL(u). \tag{21}$$

In mathematics, when something is called linear, these two properties are what it must satisfy.

Proposition 5.5. *The operator L defined above for the differential equation*

$$y'' + py' + qy = 0$$

is a linear operator.

Proof. I leave this proof as an exercise to get your hands a little dirty with linear operators. □

Our goal for this class then boils down to an inverse problem. That is, in the example above, we know that $v(x) = 0$ and we have to find $u(x)$, such that when we plug this into L, the output is zero. This whole song and dance was a prelude to proving the superposition principle above. The proof now falls out almost instantly.

Proof of Superposition Principle. We want to show that if y_1 and y_2 are two solutions to $L[y] = 0$, then $c_1y_1 + c_2y_2$ also is a solution for the same differential equation. For future reference, an expression of the form $c_1y_1 + c_2y_2$ is called a **linear combination**. The proof follows from a direct computation:

$$\begin{aligned} L[c_1y_1 + c_2y_2] &= L[c_1y_1] + L[c_2y_2] \\ &= c_1L[y_1] + c_2L[y_2] \\ &= 0 + 0 = 0 \end{aligned}$$

where the first two equalities follow from the definition of a linear operator and the last equality follows from the assumption that y_1 and y_2 solve $L[y] = 0$. □

5.2 The initial value problem and the Wronskian

So far, we have not included any initial conditions. For a second order differential equation, there are two initial conditions:

$$y(0) = a \quad \text{and} \quad y'(0) = b.$$

The solution satisfying these two initial condition is tantamount to solving for the constants c_1 and c_2 we have been writing as part of the solution. We present this result as a theorem.

Theorem 5.6. *For the second order linear homogeneous differential equation*

$$y'' + py' + qy = 0$$

with initial conditions

$$\begin{aligned} y(x_0) &= a \\ y'(x_0) &= b \end{aligned}$$

the solution set $c_1y_1 + c_2y_2$ is enough to satisfy the initial condition.

Proof. If $y = c_1y_1 + c_2y_2$, then satisfying the initial conditions implies the following system:

$$\begin{aligned} a &= c_1y_1(x_0) + c_2y_2(x_0) \\ b &= c_1y_1'(x_0) + c_2y_2'(x_0) \end{aligned}$$

The goal is to find c_1 and c_2. While the above system might look a bit obscure, this is just a linear algebra problem from earlier in the term. When do we know that there is a unique solution to the above system? The answer is that the corresponding augmented matrix must be invertible. As we recall, a matrix is invertible when its determinant is nonzero. This is where the *Wronskian* arises. If the Wronskian is nonzero, then the solutions y_1 and y_2 are linearly independent. □

We have the following result, pertaining to the Wronskian.

Theorem 5.7. *If y_1 and y_2 are solutions to the differential equation*

$$y'' + p(x)y' + q(x)y = 0,$$

then either

1. *$W(y_1, y_2) = 0$ for all x, or*
2. *$W(y_1, y_2)$ is never zero.*

5.3 Normalized solutions

When we write the combination of solutions $\{c_1y_1 + c_2y_2\}$, there is nothing sacrosanct about the solutions y_1 and y_2. Suppose that we consider the set of solutions

$$\{c_1'u_1 + c_2'u_2\}$$

where u_1 and u_2 are any other pair of solutions. Why does this matter? The solutions that we find might not be the *best* solutions. We present the following way to pick a solution.

Definition 5.8. Solutions Y_1 and Y_2 to the differential equation

$$y'' + p(x)y' + q(x)y = 0$$

are called **normalized solutions** if they satisfy the conditions

1. $Y_1(0) = 1$ and $Y_1'(0) = 0$,
2. $Y_2(0) = 0$ and $Y_2'(0) = 1$.

Lets consider a basic example of normalized solutions.

Example 5.9. Consider the differential equation

$$y'' - y = 0.$$

The general solution we obtain from the characteristic polynomial is

$$y = c_1e^x + c_2e^{-x}.$$

This solution, however, is not normalized. To find the normalized solution, we note that

$$\begin{aligned} y &= c_1e^x + c_2e^{-x} \\ y &= c_1e^x - c_2e^{-x}. \end{aligned}$$

At the point $x = 0$, we have the following system

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 0. \end{aligned}$$

It is easy to see that $c_1 = c_2 = \dfrac{1}{2}$ satisfy the system. So, $Y_1 = \dfrac{e^x + e^{-x}}{2}$. In a similar way, $Y_2 = \dfrac{e^x - e^{-x}}{2}$. These are the normalized solutions to the above differential equations. A certain number of you might realize that the normalized solutions correspond to the hyperbolic trig functions. This is one of the most natural ways in which these functions enter into mathematics.

What is so good about these solutions? If Y_1 and Y_2 are normalized, the solution to the initial value problem

$$\begin{aligned} y'' + p(x)y' + q(x)y &= 0 \\ y(0) &= y_0 \\ y'(0) &= y_0'. \end{aligned}$$

is of the form

$$y = y_0 Y_1 + y_0' Y_2.$$

In other words, you can immediately write down the solution to the initial value problem. This is why most engineers prefer this method of writing solutions. It is easy to check this is the case and I leave it to you to prove this.

5.4 Form of solutions

To understand why the two solutions are **all** the solutions, we need to invoke an important theorem. The proof is beyond the scope of Math 2400. (It is fun, though).

Theorem 5.10. *If the functions $p(x)$ and $q(x)$ are continuous functions on an interval I. Then, the differential equation*

$$y'' + p(x)y' + q(x)y = 0$$

has one ***and only one*** *solution on I such that $y(0) = y_0$ and $y'(0) = y_0'$.*

What we want is *all* the solutions to the differential equation above. The claim is that the set of normalized solutions

$$\{c_1 Y_1 + c_2 Y_2\}$$

are *all* the solutions. To see this, consider an arbitrary solution $u(x)$ to the differential equation with values $u(0) = u_0$ and $u'(0) = u_0'$. Then, from the uniqueness part of the theorem above, the function

$$u_0 Y_1 + u_0' Y_2$$

must be equal to u. So, u was not anything but a form of the solution we already found. This argument is subtle and might be the first time you have seen something like this. Take some time to digest this.

6 Inhomogeneous Equations

Up to this point, we have studied **homogeneous** linear differential equations, where the right-hand side is zero. We now turn to the more general case of **inhomogeneous** equations, which take the form

$$L[y] = g(x),$$

where L is a linear differential operator and $g(x)$ is a given (nonzero) function. We often call $g(x)$ as the **input** and the corresponding solution $y(x)$ as the **response**.

Definition 6.1. A function $y_p(x)$ is called a **particular solution** of the inhomogeneous equation if

$$L[y_p] = g(x).$$

In contrast, solutions to the associated homogeneous equation

$$L[y] = 0$$

are called **homogeneous solutions**. The key goal is to describe *all* solutions to the inhomogeneous equation.

The structure of solutions follows directly from the linearity of the operator L. Recall that linearity means

$$L[c_1y_1 + c_2y_2] = c_1L[y_1] + c_2L[y_2].$$

for functions y_1 and y_2 and constants c_1 and c_2.

Theorem 6.2. *Suppose that y_p is a particular solution with $L[y_p] = g(x)$ and y_h is any solution of the homogeneous equation with $L[y_h] = 0$. Then, the general solution of the differential equation $L[y] = g(x)$ is given by*

$$y(x) = y_p(x) + y_h(x) \tag{22}$$

Proof. We first have to prove that equation (22) actually solves the differential equation. It follows that

$$\begin{aligned} L[y_h + y_p] &= L[y_h] + L[y_p] \\ &= L[c_1y_1 + c_2y_2] + L[y_p] \\ &= c_1L[y_1] + c_2L[y_2] + L[y_p] \\ &= 0 + 0 + g(x). \end{aligned}$$

□

This observation leads to the following fundamental result:

Theorem 6.3. *Every solution of $L[y] = g(x)$ can be written in the form*

$$y(x) = y_p(x) + y_h(x),$$

where y_p is a particular solution and y_h is a solution of the homogeneous equation.

Proof. Let y be any solution of the inhomogeneous equation. Then

$$L[y] = g(x), \quad L[y_p] = g(x).$$

Subtracting the two equations and using the linearity of L yields that

$$L[y - y_p] = L[y] - L[y_p] = g(x) - g(x) = 0.$$

Thus, $y - y_p$ is a solution of the homogeneous equation, meaning

$$y - y_p = y_h$$

which yields

$$y = y_p + y_h.$$

□

Proposition 6.4. *Particular solutions are* not unique. *However, they are unique up to the addition of a homogeneous solution.*

Proof. Suppose y_p and $\tilde{y}_p$ are both particular solutions:

$$L[y_p] = g(x), \quad L[\tilde{y}_p] = g(x).$$

Subtracting the two equations and using the linearity of L yields that

$$L[y_p - \tilde{y}_p] = 0,$$

so $y_p - \tilde{y}_p$ is a homogeneous solution. Thus, any two particular solutions differ by a solution of the homogeneous equation. □

While there are infinitely many particular solutions, they all differ by homogeneous solutions. Consequently, the expression

$$y = y_p + y_h$$

captures *all possible solutions* without ambiguity.

6.1 Exponential input

One of the most important and useful cases of inhomogeneous equations occurs when the forcing function is an exponential. That is, we consider equations of the form

$$L[y] = e^{\lambda x},$$

where $\lambda \in \mathbb{C}$. The key reason exponentials are so useful is that they interact very simply with differentiation. Indeed, if $y = e^{\lambda x}$, then

$$\frac{d}{dx} e^{\lambda x} = \lambda e^{\lambda x}, \quad \frac{d^n}{dx^n} e^{\lambda x} = \lambda^n e^{\lambda x}.$$

Additionally, every periodic function can be decomposed to an infinite sum of complex exponentials– this is the study of Fourier series. This property allows us to reduce differential equations with exponential inputs to algebraic computations.

6.1.1 The Exponential Response Formula

Theorem 6.5. *Consider the linear operator*

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y,$$

whose characteristic polynomial is given as

$$p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0.$$

If $p(\lambda) \neq 0$*, then a particular solution to*

$$L[y] = e^{\lambda x}$$

is given by

$$y_p(x) = \frac{1}{p(\lambda)} e^{\lambda x}. \tag{23}$$

Proof. Let $y = e^{\lambda x}$. Then, as noted above,

$$y^{(k)} = \lambda^k e^{\lambda x}.$$

Substituting into the operator L, we obtain

$$L[e^{\lambda x}] = a_n \lambda^n e^{\lambda x} + a_{n-1}\lambda^{n-1} e^{\lambda x} + \cdots + a_1 \lambda e^{\lambda x} + a_0 e^{\lambda x}.$$

Factoring out $e^{\lambda x}$ gives

$$L[e^{\lambda x}] = \left(a_n \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_0\right) e^{\lambda x}.$$

Thus,

$$L[e^{\lambda x}] = p(\lambda)\, e^{\lambda x}.$$

Now define

$$y_p(x) = \frac{1}{p(\lambda)} e^{\lambda x}.$$

By linearity,

$$L[y_p] = \frac{1}{p(\lambda)} L[e^{\lambda x}] = \frac{1}{p(\lambda)} \cdot p(\lambda) e^{\lambda x} = e^{\lambda x}.$$

□

Example 6.6. Solve the inhomogeneous differential equation

$$y'' - 3y' + 2y = e^{3x}.$$

Solution. The characteristic polynomial at the value $r = 3$ is given by

$$p(3) = 9 - 9 + 2 = 2 \neq 0.$$

Thus, by the exponential response formula,

$$y_p(x) = \frac{1}{2} e^{3x}.$$

□

6.1.2 The Resonance Response Formula

The exponential response formula works well when the forcing term $e^{\lambda x}$ is *not* already part of the homogeneous solution. The interesting case is when it *is* part of the homogeneous solution. In that case, the naive guess

$$y_p = A e^{\lambda x}$$

fails, because applying the differential operator produces zero instead of the desired forcing term. This is called **resonance**.

Resonance happens when the forcing term looks like a solution of the homogeneous equation. In that case, the operator L annihilates the naive guess, so we must multiply by enough powers of x to get something new.

The correct particular solution is obtained by multiplying by a suitable power of x. The result is called the **resonance response formula**.

Theorem 6.7 (Resonance response formula)**.** *Let*

$$L = p(D)$$

be a constant-coefficient linear differential operator, where $D = \frac{d}{dx}$ *and*

$$p(r) = a_n r^n + a_{n-1} r^{n-1} + \cdots + a_1 r + a_0.$$

Suppose the forcing term is $e^{\lambda x}$*, and suppose* λ *is a root of* $p(r)$ *of multiplicity* $s \geq 1$*. Then a particular solution of*

$$L[y] = e^{\lambda x}$$

has the form

$$y_p(x) = \frac{x^s e^{\lambda x}}{p^{(s)}(\lambda)},$$

In words, if $e^{\lambda x}$ resonates with the homogeneous equation, then we multiply by x^s, where s is the multiplicity of the root λ. To prove the the resonance response formula, we need to understand how to apply linear operators to objects like $e^{\lambda x} u(x)$. Thankfully, such a tool exists. The key tool behind this formula is the **exponential shift law**.

Theorem 6.8 (Exponential Shift Law)**.** *If* $p(D)$ *is a polynomial in the differential operator* D*, then*

$$p(D)\big(e^{\lambda x} u(x)\big) = e^{\lambda x} p(D + \lambda) u(x),$$

for any sufficiently differentiable function u*.*

Proof. We first prove the identity for the first derivative. Using the product rule,

$$D\big(e^{\lambda x} u(x)\big) = \frac{d}{dx}\big(e^{\lambda x} u(x)\big) = \lambda e^{\lambda x} u(x) + e^{\lambda x} u'(x).$$

Factoring out $e^{\lambda x}$ gives

$$D\big(e^{\lambda x} u(x)\big) = e^{\lambda x}(D + \lambda) u(x).$$

Now apply D repeatedly. Since D acts on $e^{\lambda x} u(x)$ by turning into $D + \lambda$ on the factor $u(x)$, we get

$$D^2\big(e^{\lambda x} u\big) = D\Big(e^{\lambda x}(D + \lambda) u\Big) = e^{\lambda x}(D + \lambda)^2 u,$$

and similarly, by induction,

$$D^n\big(e^{\lambda x} u\big) = e^{\lambda x}(D + \lambda)^n u.$$

Because any polynomial in D is a linear combination of powers of D, the same identity holds for $p(D)$:

$$p(D)\big(e^{\lambda x} u\big) = e^{\lambda x} p(D + \lambda) u.$$

□

We now have the tools to prove the resonance response formula.

Proof of resonance response formula. Suppose that the characteristic polynomial can be written as

$$p(r) = (r-\lambda)^s q(r), \qquad q(\lambda) \neq 0.$$

We try a particular solution of the form

$$y_p(x) = Ax^s e^{\lambda x},$$

where A is a constant to be determined. Applying the operator $L = p(D)$ and using the shift law gives

$$L[y_p] = A\, p(D)\big(x^s e^{\lambda x}\big) = A\, e^{\lambda x} p(D+\lambda) x^s.$$

Since

$$p(D+\lambda) = D^s q(D+\lambda),$$

we have

$$p(D+\lambda)x^s = D^s\big(q(D+\lambda)x^s\big).$$

Now $q(D+\lambda)x^s$ is a polynomial of degree at most s, and its top-degree term is $q(\lambda)x^s$. Therefore, when we apply D^s, all lower-degree terms disappear, and the result is

$$D^s\big(q(D+\lambda)x^s\big) = s!\, q(\lambda).$$

Hence

$$L[y_p] = A\, e^{\lambda x}\, s!\, q(\lambda).$$

To make this equal to $e^{\lambda x}$, we choose

$$A = \frac{1}{s!\, q(\lambda)}.$$

Therefore,

$$y_p(x) = \frac{x^s e^{\lambda x}}{s!\, q(\lambda)} = \frac{x^2 e^{\lambda x}}{p^{(s)}(\lambda)}$$

is a particular solution, as claimed. □

Resonance happens when the forcing term looks like a solution of the homogeneous equation. In that case, the operator L annihilates the naive guess, so we must multiply by enough powers of x to get something new.

Example 6.9. Solve the inhomogeneous differential equation

$$y'' - 2y' + y = e^x.$$

Solution. The characteristic polynomial is

$$p(r) = r^2 - 2r + 1 = (r-1)^2.$$

So $\lambda = 1$ is a root of multiplicity $s = 2$, and we calculate $p''(1) = 2$. The resonance response formula gives

$$y_p(x) = \frac{1}{2}x^2 e^x.$$

□

For the problems you will encounter in this class, we have the following three results:

- If λ is not a root of $p(r)$, then $s = 0$ and the formula reduces to

$$y_p = \frac{1}{p(\lambda)} e^{\lambda x}.$$

- If λ is a simple root, then $s = 1$ and we use

$$y_p = \frac{x e^{\lambda x}}{p'(\lambda)}.$$

- If λ is a root of multiplicity 2, then we use

$$y_p = \frac{x^2 e^{\lambda x}}{p''(\lambda)}.$$

6.2 Sinusoidal input

We now consider inhomogeneous equations where the forcing term is sinusoidal:

$$L[y] = \cos(\omega x) \quad \text{or} \quad L[y] = \sin(\omega x).$$

At first glance, these inputs appear different from exponentials. However, using complex exponentials, we can reduce this case to the one we already understand. Recall Euler's formulas:

$$\cos(\omega x) = \Re\big(e^{i\omega x}\big), \quad \sin(\omega x) = \Im\big(e^{i\omega x}\big).$$

Thus, instead of solving

$$L[y] = \cos(\omega x),$$

we consider the *complex* equation

$$L[z] = e^{i\omega x},$$

and then take the real or imaginary part of the solution. This can be seen as follows: If $L[z] = e^{i\omega x}$ and $z = u + iv$, then

$$L[u] = \cos(\omega x), \quad L[v] = \sin(\omega x).$$

Let $L = p(D)$ with characteristic polynomial $p(r)$. Then

$$L[e^{i\omega x}] = p(i\omega)\, e^{i\omega x}.$$

If $p(i\omega) \neq 0$, the exponential response formula gives

$$z_p(x) = \frac{1}{p(i\omega)} e^{i\omega x}.$$

To obtain real-valued solutions, we take real and imaginary parts:

$$y_p^{(\cos)}(x) = \Re\left(\frac{1}{p(i\omega)} e^{i\omega x}\right), \quad y_p^{(\sin)}(x) = \Im\left(\frac{1}{p(i\omega)} e^{i\omega x}\right).$$

In practice, we write

$$\frac{1}{p(i\omega)} = A + iB,$$

where $A, B \in \mathbb{R}$. Then

$$\frac{1}{p(i\omega)} e^{i\omega x} = (A + iB)(\cos(\omega x) + i\sin(\omega x)).$$

Multiplying,

$$= (A\cos(\omega x) - B\sin(\omega x)) + i(A\sin(\omega x) + B\cos(\omega x)).$$

Thus,

$$y_p^{(\cos)}(x) = A\cos(\omega x) - B\sin(\omega x),$$
$$y_p^{(\sin)}(x) = A\sin(\omega x) + B\cos(\omega x).$$

Example 6.10. Solve the following inhomogeneous differential equation

$$y'' + 2y' + 2y = \cos x.$$

Solution. The characteristic polynomial is given by

$$p(r) = r^2 + 2r + 2,$$

and we calculate $p(i)$ as

$$p(i) = (i)^2 + 2i + 2 = -1 + 2i + 2 = 1 + 2i.$$

Thus,

$$\frac{1}{p(i)} = \frac{1}{1+2i} = \frac{1}{1+2i} \cdot \frac{1-2i}{1-2i} = \frac{1-2i}{5}.$$

So

$$z_p(x) = \frac{1-2i}{5} e^{ix} = \frac{1-2i}{5}(\cos(x) + i\sin(x)).$$

Taking the real part:

$$y_p(x) = \frac{1}{5}(\cos x + 2\sin x).$$

□

Example 6.11. Solve the inhomogeneous differential equation

$$y'' + y = \sin x.$$

Solution. The characteristic polynomial is

$$p(r) = r^2 + 1.$$

Evaluating at $r = i$ gives

$$p(i) = -1 + 1 = 0,$$

so we are in the **resonance case**. This means the standard exponential response formula fails, since e^{ix} is already a solution of the homogeneous equation. Since we have a sinusoidal input, we consider the equation

$$L[z] = e^{ix}.$$

Since $\lambda = i$ is a root of multiplicity 1, the resonance response formula tells us to try

$$z_p(x) = \frac{xe^{ix}}{p'(i)} = \frac{xe^{ix}}{2i} = -\frac{i}{2}xe^{ix}$$

We compute the imaginary part via

$$z_p(x) = -\frac{i}{2}x(\cos x + i\sin x) = \frac{x}{2}\sin x - i\frac{x}{2}\cos x.$$

and hence

$$y_p(x) = -\frac{x}{2}\cos x.$$

□

This method shows that sinusoidal inputs are naturally handled using complex exponentials. The operator L acts on $e^{i\omega x}$ just like it does on $e^{\lambda x}$, and the real and imaginary parts recover cosine and sine. To solve $L[y] = \cos(\omega x)$ or $L[y] = \sin(\omega x)$:

1. Solve $L[z] = e^{i\omega x}$ using the exponential response formula,
2. Write the result in the form $(A + iB)e^{i\omega x}$,
3. Take the real or imaginary part to obtain a real-valued particular solution.

6.3 Polynomial inputs

We now consider inhomogeneous equations where the forcing term is a polynomial:

$$L[y] = P_n(x),$$

where $P_n(x)$ is a polynomial of degree n. Unlike exponential or sinusoidal inputs, there is no direct "response formula." Instead, we use the **method of undetermined coefficients**, which relies on guessing the form of a solution and determining the coefficients by substitution. The key observation is that derivatives of polynomials are again polynomials. Therefore, when a linear differential operator L is applied to a polynomial, the result is still a polynomial. This suggests that we look for a particular solution of the same general form:

$$y_p(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0,$$

where the coefficients $a_0, \ldots, a_n$ are unknown.

We then substitute y_p into the equation $L[y] = P_n(x)$ and solve for the coefficients by matching powers of x. This method is best illustrated by computing a few examples.

Example 6.12. Solve the inhomogeneous differential equation

$$y'' - y = x^2.$$

Solution. We try a particular solution of the form

$$y_p(x) = ax^2 + bx + c.$$

Compute derivatives:

$$y_p' = 2ax + b, \quad y_p'' = 2a.$$

Substitute into the differential equation:

$$y_p'' - y_p = 2a - (ax^2 + bx + c).$$

This simplifies to

$$-ax^2 - bx + (2a - c).$$

We want this to equal x^2, so we match coefficients:

$$-a = 1, \quad -b = 0, \quad 2a - c = 0.$$

Solving:

$$a = -1, \quad b = 0, \quad c = -2.$$

Thus,

$$y_p(x) = -x^2 - 2.$$

□

Example 6.13. Solve the inhomogeneous differential equation

$$y'' - 2y' + y = 1.$$

Solution. The characteristic equation is

$$(r - 1)^2 = 0,$$

so the homogeneous solutions are e^x and xe^x. We first try a constant:

$$y_p = A.$$

Substituting gives

$$0 - 0 + A = 1.$$

So $A = 1$ works, and there is no conflict with the homogeneous solution. Thus,

$$y_p = 1.$$

□

As with exponentials, we must be careful if the polynomial we guess overlaps with the homogeneous solution. If any term in your guess is already a solution to the homogeneous equation, multiply the entire guess by x (or a higher power of x if needed). In general, to solve $L[y] = P_n(x)$:

1. Guess a polynomial of the same degree as $P_n(x)$,
2. If the guess overlaps with the homogeneous solution, multiply by x until it no longer does,
3. Substitute into the differential equation,
4. Match coefficients to solve for the unknown constants.

6.4 Variation of Parameters

The method of **variation of parameters** is a general technique for finding a particular solution to a second-order linear inhomogeneous equation. Unlike the method of undetermined coefficients, it does not require the forcing term to have any special form. As a result, it is much more flexible, although it often involves more computation. We consider an equation in standard form:

$$y'' + p(x)y' + q(x)y = g(x).$$

Suppose that y_1 and y_2 are linearly independent solutions of the associated homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

Then the general homogeneous solution is

$$y_h(x) = C_1 y_1(x) + C_2 y_2(x).$$

The idea behind variation of parameters is to look for a particular solution in the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where u_1 and u_2 are functions to be determined. The constants from the homogeneous solution are replaced by functions, hence the name "variation of parameters." We now derive formulas for u_1 and u_2. Start with the ansatz

$$y_p = u_1 y_1 + u_2 y_2.$$

Differentiating gives

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

At this point we impose the auxiliary condition

$$u_1' y_1 + u_2' y_2 = 0.$$

This is not an additional assumption about the solution; it is a convenient choice that simplifies the computation. With this condition, the first derivative becomes

$$y_p' = u_1 y_1' + u_2 y_2'.$$

Differentiating once more,

$$y_p'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''.$$

Now substitute y_p, y_p', and y_p'' into the differential equation:

$$y_p'' + p(x)y_p' + q(x)y_p = g(x).$$

This gives

$$\bigl(u_1'y_1' + u_2'y_2'\bigr) + u_1\bigl(y_1'' + py_1' + qy_1\bigr) + u_2\bigl(y_2'' + py_2' + qy_2\bigr) = g(x).$$

Because y_1 and y_2 solve the homogeneous equation, the terms in parentheses vanish:

$$y_1'' + py_1' + qy_1 = 0, \qquad y_2'' + py_2' + qy_2 = 0.$$

So we are left with

$$u_1'y_1' + u_2'y_2' = g(x).$$

Together with the auxiliary condition

$$u_1'y_1 + u_2'y_2 = 0,$$

we obtain a 2×2 linear system for u_1' and u_2':

$$\begin{cases} u_1'y_1 + u_2'y_2 = 0, \\ u_1'y_1' + u_2'y_2' = g(x). \end{cases}$$

The determinant of this system is the Wronskian

$$W(y_1, y_2)(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_1'y_2.$$

Since y_1 and y_2 are linearly independent, their Wronskian is nonzero on the interval of interest. Using Cramer's rule, we get

$$u_1' = \frac{\begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix}}{W} = -\frac{y_2 g}{W},$$

and

$$u_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix}}{W(y_1, y_2)} = \frac{y_1 g}{W(y_1, y_2)}.$$

Thus the formulas for the derivatives of the unknown functions are

$$u_1'(x) = -\frac{y_2(x)g(x)}{W(y_1, y_2)(x)}, \qquad u_2'(x) = \frac{y_1(x)g(x)}{W(y_1, y_2)(x)}.$$

Integrating gives

$$\boxed{u_1(x) = -\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}\,dx, \qquad u_2(x) = \int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}\,dx.}$$

Any constants of integration may be ignored, since they only contribute terms already contained in the homogeneous solution. Therefore a particular solution is

$$y_p(x) = -y_1(x)\int \frac{y_2(x)g(x)}{W(y_1,y_2)(x)}\,dx + y_2(x)\int \frac{y_1(x)g(x)}{W(y_1,y_2)(x)}\,dx.$$

This is the **variation of parameters formula** for second-order linear equations.

The key idea is that the homogeneous solutions already describe the natural behavior of the differential operator. The forcing term $g(x)$ changes the solution by making the coefficients "move" instead of staying constant. The auxiliary condition

$$u_1'y_1 + u_2'y_2 = 0$$

is chosen so that the resulting algebra becomes manageable. Without it, differentiating $u_1y_1 + u_2y_2$ would produce too many unknown terms.

Another important point is that this method is completely general: it does not depend on whether $g(x)$ is a polynomial, exponential, trigonometric function, or something more complicated. As long as the homogeneous solutions are known, variation of parameters can be used.

Example 6.14. Solve the inhomogeneous differential equation

$$y'' + y = \tan x.$$

Solution. The homogeneous equation is

$$y'' + y = 0,$$

with fundamental solutions

$$y_1(x) = \cos x, \qquad y_2(x) = \sin x.$$

Calculating the Wronskian, we find

$$W(\sin(x), \cos(x)) = \cos x \cdot \cos x - (-\sin x)(\sin x) = \cos^2 x + \sin^2 x = 1.$$

Using variation of parameters,

$$u_1' = -y_2 g = -\sin x \tan x = -\frac{\sin^2 x}{\cos x},$$

$$u_2' = y_1 g = \cos x \tan x = \sin x.$$

Integrating,

$$u_2 = \int \sin x\,dx = -\cos x,$$

and

$$u_1 = -\int \frac{\sin^2 x}{\cos x}\,dx = -\int \frac{1-\cos^2 x}{\cos x}\,dx = -\int (\sec x - \cos x)\,dx.$$

So

$$u_1 = -\ln|\sec x + \tan x| + \sin x.$$

A particular solution is therefore

$$y_p = u_1 \cos x + u_2 \sin x.$$

After simplifying, one possible form is

$$y_p(x) = -\cos x \ln|\sec x + \tan x|.$$

□

In general, for an equation of the form

$$y'' + p(x)y' + q(x)y = g(x),$$

if y_1 and y_2 are linearly independent solutions of the homogeneous equation, then a particular solution is obtained by setting

$$y_p = u_1 y_1 + u_2 y_2$$

with

$$u_1' = -\frac{y_2 g}{W(y_1, y_2)}, \qquad u_2' = \frac{y_1 g}{W(y_1, y_2)},$$

where

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2.$$

7 Applications to engineering

7.1 The Mass–Spring–Dashpot System

One of the most important applications of second-order differential equations is the **mass–spring–dashpot system**. This model describes the motion of an object attached to a spring, with resistance coming from a damper or dashpot. It is a standard model for vibrations, oscillations, and mechanical motion.

Consider a mass m attached to a spring with spring constant k and a dashpot that provides damping with coefficient γ. Let $x(t)$ denote the displacement of the mass from its equilibrium position at time t.

We assume that:

- the spring obeys Hooke's law,
- the dashpot exerts a force proportional to velocity,
- motion takes place along a line,

- positive displacement is measured to the right.

We may also allow an external forcing function $F(t)$ acting on the mass. Newton's second law says that

$$mx''(t) = (\text{sum of forces}).$$

There are three main forces in the system. By Hooke's law, the spring exerts a restoring force proportional to displacement:

$$F_{\text{spring}} = -kx.$$

The negative sign indicates that the force points back toward equilibrium. The dashpot resists motion, and its force is proportional to velocity:

$$F_{\text{damping}} = -\gamma x'.$$

Again, the negative sign shows that the force opposes motion. If an outside force is applied, we denote it by

$$F(t).$$

Adding these forces together gives

$$mx'' = -\gamma x' - kx + F(t).$$

Rearranging,

$$mx'' + \gamma x' + kx = F(t).$$

This is the standard **mass–spring–dashpot equation**.

7.1.1 The Homogeneous System

If there is no external force, then $F(t) = 0$ and the equation becomes

$$mx'' + \gamma x' + kx = 0.$$

This is a second-order linear homogeneous differential equation with constant coefficients. To solve it, we form the characteristic equation

$$mr^2 + \gamma r + k = 0.$$

The behavior of the system depends on the roots of this quadratic. If

$$\gamma^2 > 4mk,$$

then the characteristic equation has two distinct real roots. In this case the motion returns to equilibrium without oscillating. The solution has the form

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t},$$

where r_1 and r_2 are real and negative. This is called **overdamped** motion. If

$$\gamma^2 = 4mk,$$

then the characteristic equation has a repeated real root. The solution has the form

$$x(t) = (C_1 + C_2 t)e^{rt}.$$

This is called **critically damped** motion. Critical damping is the fastest return to equilibrium without oscillation. If

$$\gamma^2 < 4mk,$$

then the roots are complex:

$$r = \alpha \pm i\beta.$$

In this case,

$$x(t) = e^{\alpha t}\big(C_1 \cos(\beta t) + C_2 \sin(\beta t)\big).$$

This is called **underdamped** motion. The solution oscillates while gradually losing energy due to damping. Each parameter has the physical meaning:

- m is the mass of the object,
- k is the stiffness of the spring,
- γ measures the strength of the damping,
- $F(t)$ is the external forcing.

Larger values of k make the spring stiffer, so the system oscillates more rapidly. Larger values of γ increase resistance and reduce oscillation. Larger values of m make the system harder to accelerate.

7.1.2 Forced Motion

When $F(t) \neq 0$, the equation becomes inhomogeneous:

$$mx'' + \gamma x' + kx = F(t).$$

The general solution is

$$x(t) = x_h(t) + x_p(t),$$

where x_h is the homogeneous solution and x_p is a particular solution. The homogeneous solution describes the natural motion of the system, while the particular solution describes the effect of the forcing term. If the forcing function has a frequency close to the natural frequency of the system, the amplitude of the oscillation can become large. This phenomenon is called **resonance**. For example, if

$$F(t) = F_0 \cos(\omega t),$$

then the motion may be especially large when ω matches the natural frequency of the system. Damping prevents the amplitude from growing without bound, but resonance can still produce dramatic oscillations.

Example 7.1. Consider the mass–spring–dashpot system

$$2x'' + 3x' + 2x = 0.$$

Here,

$$m = 2, \quad \gamma = 3, \quad k = 2.$$

The characteristic equation is

$$2r^2 + 3r + 2 = 0.$$

Solving,

$$r = \frac{-3 \pm \sqrt{9 - 16}}{4} = \frac{-3 \pm i\sqrt{7}}{4}.$$

Thus, the solution is

$$x(t) = e^{-3t/4}\left(C_1 \cos\left(\frac{\sqrt{7}}{4}t\right) + C_2 \sin\left(\frac{\sqrt{7}}{4}t\right)\right).$$

This is an underdamped system: the motion oscillates while gradually decaying to zero.

Example 7.2. Consider

$$x'' + x = \cos t.$$

This is a forced oscillator with no damping. The homogeneous equation is

$$x'' + x = 0,$$

whose solutions are

$$x_h(t) = C_1 \cos t + C_2 \sin t.$$

Since the forcing term $\cos t$ matches one of the homogeneous solutions, resonance occurs. Therefore, a particular solution must include a factor of t. The result is

$$x_p(t) = \frac{1}{2} t \sin t.$$

So, the general solution is

$$x(t) = C_1 \cos t + C_2 \sin t + \frac{1}{2} t \sin t.$$

7.2 Second-Order Circuits (RLC Circuits)

Another fundamental application of differential equations arises in electrical circuits. In particular, **RLC circuits**—which contain a resistor, inductor, and capacitor—are governed by second-order linear differential equations that closely parallel the mass–spring–dashpot system. We consider a series circuit consisting of:

- a resistor with resistance R,
- an inductor with inductance L,

- a capacitor with capacitance C,
- a voltage source $E(t)$.

Let $q(t)$ denote the charge on the capacitor at time t, and let $i(t) = q'(t)$ be the current in the circuit. The derivation of the differential equation is based on **Kirchhoff's Voltage Law (KVL)**, which states:

Proposition 7.3. *The sum of the voltage drops across each component equals the applied voltage.*

Each component contributes a voltage drop:

Resistor. By Ohm's law,

$$V_R = Ri = Rq'.$$

Inductor. The voltage across an inductor is proportional to the rate of change of current:

$$V_L = L\frac{di}{dt} = Lq''.$$

Capacitor. The voltage across a capacitor is proportional to the charge:

$$V_C = \frac{1}{C}q.$$

Applying Kirchhoff's law,

$$V_L + V_R + V_C = E(t),$$

we obtain

$$Lq'' + Rq' + \frac{1}{C}q = E(t).$$

This is the standard **RLC circuit equation**.

7.2.1 The Homogeneous Circuit

If there is no external voltage ($E(t) = 0$), the equation becomes

$$Lq'' + Rq' + \frac{1}{C}q = 0.$$

The characteristic equation is

$$Lr^2 + Rr + \frac{1}{C} = 0.$$

As in the mechanical case, the behavior depends on the discriminant:

$$R^2 - \frac{4L}{C}.$$

If

$$R^2 > \frac{4L}{C},$$

the roots are real and distinct. The charge decays to zero without oscillation:

$$q(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}.$$

If

$$R^2 = \frac{4L}{C},$$

there is a repeated root, and

$$q(t) = (C_1 + C_2 t)e^{rt}.$$

If

$$R^2 < \frac{4L}{C},$$

the roots are complex:

$$r = \alpha \pm i\beta,$$

and

$$q(t) = e^{\alpha t}\big(C_1 \cos(\beta t) + C_2 \sin(\beta t)\big).$$

In this case, the charge oscillates while gradually decaying due to resistance. Recall that current is

$$i(t) = q'(t).$$

Thus, once we find $q(t)$, the current can be obtained by differentiation.

7.2.2 Forced Circuits

When a voltage source is present, the equation becomes

$$Lq'' + Rq' + \frac{1}{C}q = E(t).$$

The general solution is

$$q(t) = q_h(t) + q_p(t),$$

where:

- q_h describes the natural response of the circuit,
- q_p describes the forced response due to $E(t)$.

A common input is a sinusoidal voltage:

$$E(t) = E_0 \cos(\omega t).$$

In this case, the circuit eventually settles into a steady oscillation at the same frequency ω, after transient effects (from q_h) decay. The steady-state solution can be found using the exponential response method or variation of parameters. Resonance occurs when the frequency of the input matches the natural frequency of the circuit.For an undamped circuit ($R = 0$),

$$Lq'' + \frac{1}{C}q = E_0 \cos(\omega t),$$

the natural frequency is

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

If $\omega = \omega_0$, resonance occurs, and the amplitude of oscillation grows over time. With resistance present, resonance still occurs, but the amplitude remains bounded due to damping.

Example 7.4. Consider the circuit

$$q'' + 2q' + 2q = 0.$$

Here,

$$L = 1, \quad R = 2, \quad \frac{1}{C} = 2.$$

The characteristic equation is

$$r^2 + 2r + 2 = 0,$$

so

$$r = -1 \pm i.$$

Thus,

$$q(t) = e^{-t}\big(C_1 \cos t + C_2 \sin t\big).$$

This is an underdamped circuit: the charge oscillates while decaying over time.