HW 8 Solutions

1) a) $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

$$\lambda^2 - tr(A)\lambda + det(A) = 0$$

$$\lambda^2 - 5\lambda + 6 = 0$$

$$(\lambda - 2)(\lambda - 3) = 0$$

eigenvalues: $\lambda = 2$, $\lambda = 3$

$\lambda = 2$: $(A - 2I)\vec{\alpha}_1 = \vec{0}$

$$\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \vec{\alpha}_1 = \vec{0}$$

$$\vec{\alpha}_1 = (1, 0)^T$$

$\lambda = 3$: $(A - 3I)\vec{\alpha}_2 = \vec{0}$

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \vec{\alpha}_2 = \vec{0}$$

$$\vec{\alpha}_2 = (1, 1)^T$$

2 linearly ind. eigenvectors $\Rightarrow$ diagonalizab

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1}$$

b) $A = \begin{pmatrix} 4 & -2 \\ 1 & 1 \end{pmatrix}$

$\lambda^2 - tr(A)\lambda + \det(A) = 0$

$\lambda^2 - 5\lambda + 6 = 0$

$(\lambda - 2)(\lambda - 3) = 0$

eigenvalues: $\lambda = 2$, $\lambda = 3$

$\lambda = 2$: $(A - 2I)\vec{\alpha}_1 = \vec{0}$

$\begin{pmatrix} 2 & -2 \\ 1 & -1 \end{pmatrix} \vec{\alpha}_1 = 0 \Rightarrow \vec{\alpha}_1 = (1, 1)^T$

$\lambda = 3$: $(A - 3I)\vec{\alpha}_2 = \vec{0}$

$\begin{pmatrix} 1 & -2 \\ 1 & -2 \end{pmatrix} \vec{\alpha}_2 = \vec{0} \Rightarrow \vec{\alpha}_2 = (2, 1)^T$

2 linearly independent eigenvectors
$\Rightarrow$ A is diagonalizable

$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}^{-1}$.

2) $m = 2\,Kg$

a) $F = Kx \Rightarrow K = \frac{3}{0.5} = 6$

$F = bx' \Rightarrow b = \frac{8}{2} = 4$

$mx'' + bx' + Kx \neq \cancel{0}$

$2x'' + 4x' + 6x = 0$

$x'' + 2x' + 3x = 0$

b) $x'' + 2\zeta\omega_n x' + \omega_n^2 x$

$\omega_n^2 = 3 \Rightarrow \omega_n = \sqrt{3}$

$2\zeta\omega_n = 2 \Rightarrow \zeta = \frac{\sqrt{3}}{3} < 1$

underdamped

c) $p(r) = r^2 + 2r + 3$

$= (r+1)^2 + 2$

$p(r) = 0 \Rightarrow r = -1 \pm i\sqrt{2}$.

$x(t) = c_1 e^{-t} \cos(\sqrt{2}t) + c_2 e^{-t} \sin(\sqrt{2}t)$

$x(0) = -0.8\ ,\ x'(0) = 0.2$

$c_1 = -0.8$

$x'(t) = +0.8\left(+e^{-t}\cos(\sqrt{2}t) + \sqrt{2}e^{-t}\sin(\sqrt{2}t)\right)$

$+ c_2\left(-e^{-t}\sin(\sqrt{2}t) + \sqrt{2}e^{-t}\cos(\sqrt{2}t)\right)$

$$x'(0) = 0.8 + \sqrt{2}\, c_2 = 0.2$$

$$\Rightarrow c = -\frac{3}{5\sqrt{2}}$$

$$x(t) = e^{-t}\left(-\frac{4}{5}\cos(\sqrt{2}\,t) - \frac{3}{5\sqrt{2}}\sin(\sqrt{2}\,t)\right).$$

3) $y'' - 3y' + 2y = e^{x} + x$

$p(r) = r^2 - 3r + 2 = (r-2)(r-1)$

$y_h = c_1 e^{x} + c_2 e^{2x}$

y_{p_1} : e^{x} $\rightarrow$ repeated root

$$y_{p_1} = \frac{x e^{x}}{p'(1)} = -x e^{x}$$

y_{p_2} : $y_{p_2} = Ax + B$

$-3(A) + 2(Ax + B) = x$

$A = 1/2, \quad B = 3/4$

$y(x) = c_1 e^{x} + c_2 e^{2x} - x e^{x} + \frac{x}{2} + \frac{3}{4}$