1) a) $\{1, t, e^{3t}\}$

$$W(1,t,e^{3t}) = \begin{vmatrix} 1 & t & e^{3t} \\ 0 & 1 & 3e^{3t} \\ 0 & 0 & 9e^{3t} \end{vmatrix} = 9e^{3t} \neq 0$$

$$y = \underbrace{c_1 + c_2 t}_{\substack{\text{double root} \\ \text{at } r=0}} + \underbrace{c_3 e^{3t}}_{\substack{\text{single root} \\ \text{at } r=3}}$$

$$p(r) = r^2(r-3) = r^3 - 3r^2$$

$$y'''(t) - 3y''(t) = 0.$$

b) $\{t, e^t, e^{-t}\}$

$$W(t, e^t, e^{-t}) = \begin{vmatrix} t & e^t & e^{-t} \\ 1 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix}$$

$$= t(1+1) - e^t(0) + e^t(0) = 2t$$

$$y = \underbrace{c_1 t} + c_2 e^t + c_3 e^{-t}$$

double root but missing the constant term.

2) $y'''(x) - 2y''(x) + y'(x) = e^{2x}$

$y(0) = 0$

$y'(0) = 0$

$y''(0) = 0$

$p(r) = r^3 - 2r^2 + r = r(r^2 - 2r + 1)$

$= r(r-1)^2$

$p(r) = 0 \Rightarrow r = 0, \quad r = 1$

$y_h(x) = c_1 + c_2 e^x + c_3 x e^x$

$y_p(x) = \dfrac{e^{2x}}{p(2)} = \dfrac{e^{2x}}{2}$

$y(x) = c_1 + c_2 e^x + c_3 x e^x + \dfrac{e^{2x}}{2}$

$y(0) = c_1 + c_2 + 1/2 = 0$

$y'(0) = c_2 + c_3(0+1) + 1 = 0$

$y''(0) = c_2 + 2c_3 + 2 = 0$

$c_1 = -1/2, \quad c_2 = 0, \quad c_3 = -1$

$y(x) = -1/2 - xe^x + \dfrac{e^{2x}}{2}$

3) $y''(x) + y(x) = \tan(x)$

$$y''(x) + y(x) = 0$$

$$y_h = c_1 \cos(x) + c_2 \sin(x)$$

$$W(y_1, y_2) = \cos^2(x) + \sin^2(x) = 1$$

$$y_p = u_1 \cos(x) + u_2 \sin(x)$$

$$u_1 = -\int \frac{\tan(x)\sin(x)}{1}\,dx = -\int \frac{\sin^2(x)}{\cos(x)}\,d$$

$$= -\int \frac{1-\cos^2(x)}{\cos(x)}\,dx = \int \cos(x) - \sec(x)\,d$$

$$= \sin(x) - \log|\sec(x) + \tan(x)|$$

$$u_2 = \int \frac{\tan(x)\cos(x)}{1}\,dx = \int \sin(x)\,dx$$

$$= -\cos(x)$$

$$y_p = \sin(x)\cos(x) - \cos(x)\log|\sec(x) + \tan($$

$$- \cos(x)\sin(x)$$

$$y_p = -\cos(x)\log|\sec(x) + \tan(x)|.$$