Homework 2 Solutions

1) Solve the following system

(a) $4x-2y+6z=0$

(b) $x-y-z=0$

(c) $2x-y+3z=0$

Solution: If we subtract 2c from a, we arrive at at $0=0$. Hence, the system is inconsistent. Set $z=t$. Then we have

(a') $2x-y+3t=0$

(b') $x-y-t=0$

a'−b' : $x+4t=0 \Rightarrow x=-4t$

$-4t-y-t=0 \Rightarrow y=-5t$

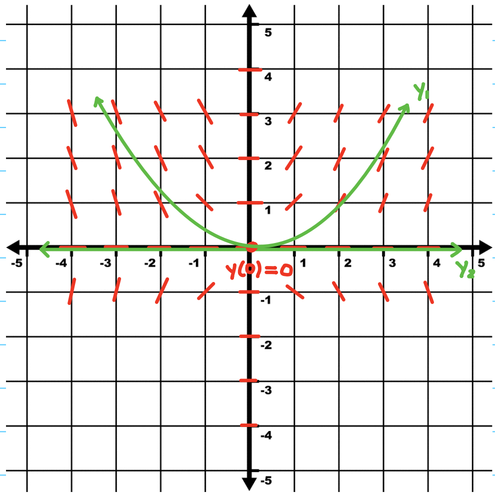
The solution is given by

$$\boxed{\{(-4t,-5t,t) \mid t \in \mathbb{R}\}}$$

2) Sketch a slope field for the ODE $\frac{dy}{dx} = xy^{1/3}$ and determine its domain. Sketch two solutions such that $y(0)=0$.

Solution: a) Slope field

x	y	$\frac{dy}{dx}$
n	0	0
0	n	0
$n>0$	1	$+n$
$n<0$	1	$-n$
$n>0$	2	$+2^{1/3}n$
$n<0$	2	$-2^{1/3}n$



b) Two solutions graphed

c) Existence and Uniqueness

$\frac{dy}{dx} = xy^{1/3} = f(x,y)$

$\partial_y f(x,y) = \frac{x}{3y^{2/3}}$

Note that $f(x,y)$ is continuous on $\mathbb{R}^2$ but $\partial_y f(x,y)$ is discontinuous on the curve $y=0$. Therefore, the solution is not unique when the initial condition $y(x_0)=0$.

3) a) $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

Solution : $\frac{dy}{dx} = \frac{x^2+y^2}{xy} \cdot \frac{1/x^2}{1/x^2} = \frac{(y/x)^2+1}{y/x}$ ← homogeneous !

Set $v = y/x \Rightarrow \frac{dy}{dx} = \frac{dv}{dx}x + v$

Substituting yields : $\frac{dv}{dx}x + v = \frac{v^2+1}{v} = v + \frac{1}{v}$

$\Rightarrow \frac{dv}{dx}x = \frac{1}{v}$

$\Rightarrow v\,dv = \frac{dx}{x}$ (separation of variables)

$\Rightarrow \frac{1}{2}v^2 = \log|x| + c$

$\Rightarrow \frac{1}{2}\frac{y^2}{x^2} = \log|x| + c$

$\Rightarrow y^2 = 2x^2(\log|x| + c)$

Solution : $\boxed{y^2 = 2x^2(\log|x| + c)}$

b) $\frac{dy}{dx} = -(2xy+y^2\cos(x))/(x^2+2y\sin(x))$

Solution: Rewrite as $(2xy+y^2\cos(x))\,dx + (x^2+2y\sin(x))\,dy = 0$

Is this ODE exact? $\partial_y M = 2x + 2y\cos(x)$

$\partial_x N = 2x + 2y\cos(x)$

Therefore, equation is exact.

1) Integrate wrt x : $\int 2xy + y^2\cos(x)\,dx = x^2y + y^2\sin(x) + f(y)$

2) Differentiate wrt y : $x^2 + 2y\sin(x) + f'(y)$

3) Compare to $N(x,y)$: $x^2 + 2y\sin(x) + f'(y) = x^2 + 2y\sin(x)$

4) Solve for $f(y)$: $f'(y) = 0 \Rightarrow f(y) = c$.

Solution: $\boxed{x^2y + y^2\sin(x) = c.}$