Key

HW 1: Sections 1.1 - 1.5

Instructions. Please take your time and explain each step thoroughly using both mathematical notation and clear English explanations. An answer without a proof will be given no credit. If you use any expressions, formulas, or results that have not been shown in class, you must prove them in your work. Make sure your work is neat, well-organized, and legible.

Problem 1. (2 pts) Give the solution to the following ODE with given initial condition $y(-4) = 0$.

$$y' = x\sqrt{x^2+9}$$

$\frac{dy}{dx} = x\sqrt{x^2+9}$

$dy = x\sqrt{x^2+9}\,dx$

$\int dy = \int x\sqrt{x^2+9}\,dx$

$u = x^2+9$

$du = 2x\,dx$

$y(x) = \frac{1}{2}\int u^{1/2}\,du = \frac{1}{3}u^{3/2} + c = \frac{1}{3}(x^2+9)^{3/2} + c$

$\boxed{y(x) = \frac{1}{3}(x^2+9)^{3/2} + c}$ (General Solution)

$y(-4) = 0$

$0 = \frac{1}{3}(16+9)^{3/2} + c$

$c = -125/3$

$\boxed{y(x) = \frac{1}{3}(x^2+9)^{3/2} - \frac{125}{3}}$

Problem 2. (4 pts) Give the general solution to the differential equation

$$y' + 4y'y = 2x.$$

$\frac{dy}{dx} + 4\frac{dy}{dx}y = 2x$

$\frac{dy}{dx}(1+4y) = 2x$

$(1+4y)dy = 2x\,dx$

$\int (1+4y)dy = \int 2x\,dx$

$\boxed{y + 2y^2 = x^2 + c}$

Problem 3. (4 pts) Give the general solution to the differential equation

$$y' - 3y(1 - \frac{1}{t}) - t^{-3}e^t = 0.$$

$\frac{dy}{dt} - 3y(1 - 1/t) - t^{-3}e^t = 0$

$\frac{dy}{dt} + y(3/t - 3) = t^{-3}e^t$

(Standard linear form)

Integrating factor: $\mu = e^{\int 3/t - 3\,dt}$

$= e^{3\log(t) - 3t}$

$= \frac{e^{3\log(t)}}{e^{3t}}$

$= \frac{t^3}{e^{3t}}$

Multiply by μ:

$\frac{t^3}{e^{3t}}\frac{dy}{dt} + y\left(\frac{3t^2}{e^{3t}} - \frac{3t^3}{e^{3t}}\right) = \frac{1}{e^{2t}}$

$= \frac{d}{dt}\left(y\frac{t^3}{e^{3t}}\right)$

$\frac{d}{dt}\left(y\frac{t^3}{e^{3t}}\right) = e^{-2t}$

$y(t)\frac{t^3}{e^{3t}} = \int e^{-2t}\,dt = \frac{-1}{2}e^{-2t} + c$

$\boxed{y(t) = \frac{-e^t}{2t^3} + \frac{ce^{3t}}{t^3}}$