Eigenvalues and Eigenvectors

Contents

1 Introduction

Linear algebra is, at its core, the study of linear transformations. Matrices provide a concrete way to represent these transformations, but in most coordinate systems their behavior can appear complicated: vectors may be stretched, rotated, reflected, and mixed together in ways that are difficult to analyze directly.

Eigenvalues and eigenvectors provide a way to cut through this complexity. They identify special directions in space along which a transformation acts in the simplest possible way. Instead of mixing coordinates or changing direction unpredictably, the transformation merely stretches or compresses along these directions. In this sense, eigenvectors reveal the hidden structure of a matrix.

This idea is powerful because it allows us to change perspective. By expressing a transformation in terms of its eigenvectors, we can often replace a complicated matrix with a much simpler one. In the best case, the transformation becomes completely decoupled into independent scaling operations. This simplification is the foundation of diagonalization and underlies many of the computational techniques we will develop.

Beyond simplification, eigenvalues and eigenvectors capture essential qualitative behavior. They describe long-term dynamics of repeated processes, determine stability in systems of differential equations, and reveal dominant patterns in data. For example, repeatedly applying a matrix to a vector often causes the result to align with a particular direction; this direction is an eigenvector, and the rate of growth or decay is governed by its eigenvalue.

These ideas appear across mathematics and its applications. In differential equations, they describe how systems evolve over time. In physics, they correspond to natural modes of vibration and energy levels. In data science, they form the basis of principal component analysis, identifying the most significant patterns in high-dimensional data.

In these notes, we will see how eigenvalues and eigenvectors provide both a conceptual lens and a practical tool: they allow us to understand the geometry of linear transformations and to compute with matrices more effectively. The goal is not only to learn how to find them, but to understand what they reveal about the underlying structure of a problem.

2 Eigenvalues

In many applications, we are interested in understanding how a matrix acts on vectors. While most vectors change both direction and magnitude under a linear transformation, there are special scalars associated with the transformation that capture its fundamental behavior. These scalars are called **eigenvalues**.

In this section, we focus only on how to *find* eigenvalues and how to recognize some of their basic properties. The corresponding eigenvectors will be introduced in the next section.

2.1 Basic definitions

Let A be an $n \times n$ matrix. We seek scalars λ that describe how A acts in a particularly simple way. The key idea is to look for values of λ such that the matrix $A - \lambda I$ fails to be invertible. Why is this important? If $A - \lambda I$ is not invertible, then the equation

$$(A - \lambda I)\mathbf{v} = 0$$

has nontrivial solutions. These values of λ turn out to be exactly the eigenvalues of A. Thus, eigenvalues arise naturally as the values for which the matrix $A - \lambda I$ becomes singular.

Definition 2.1. Let A be an $n \times n$ matrix. A scalar λ is called an **eigenvalue** of A if the matrix

$$A - \lambda I$$

is singular, equivalently if

$$\det(A - \lambda I) = 0.$$

Definition 2.2. The equation

$$\det(A - \lambda I) = 0$$

is called the **characteristic equation** of A, and the polynomial

$$p(\lambda) = \det(A - \lambda I)$$

is called the **characteristic polynomial**.

The eigenvalues of A are exactly the roots of its characteristic polynomial. The connection between eigenvalues and the characteristic equation comes from the following observation.

Suppose there is a nonzero vector $\mathbf{v}$ such that

$$A\mathbf{v} = \lambda\mathbf{v}.$$

Then

$$A\mathbf{v} - \lambda\mathbf{v} = 0,$$

so

$$(A - \lambda I)\mathbf{v} = 0.$$

Since $\mathbf{v} \neq 0$, this is a nontrivial solution of a homogeneous system. Such a system has a nontrivial solution precisely when the coefficient matrix is singular. Therefore,

$$\det(A - \lambda I) = 0.$$

2.2 Basic properties of eigenvalues

Before computing specific examples, it is useful to know a few general facts. First, we define a new quantity associated to a square matrix: the trace. This quantity is important because it provides a useful tool to check if the eigenvalues you calculates are correct.

Definition 2.3. Let $A = (a_{ij})$ be an $n \times n$ matrix. The **trace** of A, denoted $\operatorname{tr}(A)$, is the sum of its diagonal entries:

$$\operatorname{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}. \tag{1}$$

Example 2.4. Compute the trace of the matrix

$$A = \begin{pmatrix} 2 & -1 & 0 \\ 4 & 3 & 5 \\ 1 & 0 & -2 \end{pmatrix}.$$

Solution. The trace is the sum of the diagonal entries:

$$\operatorname{tr}(A) = 2 + 3 + (-2) = 3.$$

□

We are now ready to state some important properties of eigenvalues. We state them in the following proposition.

Proposition 2.5 (Basic Properties of Eigenvalues)**.** *Let A be an $n \times n$ matrix. Then:*

1. *A has at most n eigenvalues, counting multiplicity.*
2. *If A has real entries, then any nonreal eigenvalues occur in complex conjugate pairs.*
3. *If A is triangular, then its eigenvalues are exactly its diagonal entries.*
4. *The sum of the eigenvalues of A, counting multiplicity, is* $\mathrm{tr}(A)$*.*
5. *The product of the eigenvalues of A, counting multiplicity, is* $\det(A)$*.*

Proof. We prove each statement in turn.

(1) A has at most n eigenvalues, counting multiplicity.
The eigenvalues of A are the roots of the characteristic polynomial

$$p(\lambda) = \det(A - \lambda I).$$

Since $A - \lambda I$ is an $n \times n$ matrix whose entries are linear in λ, the determinant is a polynomial in λ of degree n. A polynomial of degree n has at most n roots counting multiplicity. Hence A has at most n eigenvalues, counting multiplicity.

(2) If A has real entries, nonreal eigenvalues occur in conjugate pairs.
Suppose λ is an eigenvalue of A. Then

$$p(\lambda) = \det(A - \lambda I) = 0.$$

Since A has real entries, the coefficients of $p(\lambda)$ are real. Therefore,

$$\overline{p(\lambda)} = p(\overline{\lambda}).$$

If $p(\lambda) = 0$, then taking complex conjugates gives

$$p(\overline{\lambda}) = 0.$$

So $\overline{\lambda}$ is also an eigenvalue. Thus nonreal eigenvalues appear in complex conjugate pairs.

(3) If A is triangular, then its eigenvalues are the diagonal entries. Suppose A is upper triangular. Then $A - \lambda I$ is also upper triangular, with diagonal entries

$$a_{11} - \lambda,\ a_{22} - \lambda,\ \ldots,\ a_{nn} - \lambda.$$

The determinant of a triangular matrix is the product of its diagonal entries, so

$$\det(A - \lambda I) = \prod_{j=1}^{n} (a_{jj} - \lambda).$$

Therefore the roots of the characteristic polynomial are precisely the diagonal entries $a_{11}, \ldots, a_{nn}$, counting multiplicity.(The same argument works for lower triangular matrices.)

(4) The sum of the eigenvalues equals $\operatorname{tr}(A)$. Write the characteristic polynomial as

$$p(\lambda) = \det(A - \lambda I).$$

Since $p(\lambda)$ has degree n, its leading term is

$$(-1)^n \lambda^n,$$

and its next coefficient is determined by the trace:

$$p(\lambda) = (-1)^n \lambda^n + (-1)^{n-1} \operatorname{tr}(A) \lambda^{n-1} + \cdots .$$

If the eigenvalues are $\lambda_1, \ldots, \lambda_n$ counted with multiplicity, then

$$p(\lambda) = (-1)^n (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n).$$

Expanding this product, the coefficient of λ^{n-1} is

$$(-1)^{n-1}(\lambda_1 + \cdots + \lambda_n).$$

Comparing the two expressions for the coefficient of λ^{n-1} gives

$$\lambda_1 + \cdots + \lambda_n = \operatorname{tr}(A).$$

(5) The product of the eigenvalues equals $\det(A)$. Using the same factorization,

$$p(\lambda) = (-1)^n (\lambda - \lambda_1) \cdots (\lambda - \lambda_n).$$

The constant term of this polynomial is

$$(-1)^n(-\lambda_1) \cdots (-\lambda_n) = (-1)^n(-1)^n \lambda_1 \cdots \lambda_n = \lambda_1 \cdots \lambda_n.$$

On the other hand,

$$p(0) = \det(A).$$

Therefore,

$$\lambda_1 \lambda_2 \cdots \lambda_n = \det(A).$$

□

2.3 Finding eigenvalues

At first glance, the process of finding eigenvalues may appear computational: form $A - \lambda I$, compute a determinant, and solve a polynomial equation. While this procedure is correct, it is helpful to step back and understand the *general philosophy* behind what we are doing.

The central idea is that eigenvalues are the values of λ for which the matrix $A - \lambda I$ fails to be invertible. In other words, we are looking for the values of λ where something "breaks"—where the linear transformation $A - \lambda I$ loses information.

From this point of view, finding eigenvalues is not an arbitrary procedure—it is a systematic way of detecting when a linear transformation behaves like simple scaling in some direction.

We show how to do this for the case of a 2 × 2 and 3 × 3 matrix.

2.3.1 The case of a 2×2 matrix

Consider the 2×2 matrix A defined by

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Calculating the characteristic equation yields

$$A - \lambda I = \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix},$$

so the characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I) = (a - \lambda)(d - \lambda) - bc.$$

Expanding the above polynomial gives

$$p(\lambda) = \lambda^2 - (a + d)\lambda + (ad - bc).$$

Thus, the eigenvalues are the roots of the quadratic equation

$$\lambda^2 - (a + d)\lambda + (ad - bc) = 0.$$

So for a 2×2 matrix, the eigenvalues can always be found using the quadratic formula:

$$\lambda = \frac{(a + d) \pm \sqrt{(a + d)^2 - 4(ad - bc)}}{2}.$$

No one, including me, wants to memorize this formula. Luckily, there exists an easy-to-remember formula. Notice the coefficients of the constant and linear terms in the expanded polynomial. Since $a + d$ is the trace and $ad - bc$ is the determinant, we have the following useful formula for the 2×2 case:

$$\boxed{\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0.} \tag{2}$$

We now give three representative examples illustrating the different types of eigenvalues that can arise in the 2×2 case.

Example 2.6. Find the eigenvalues of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Solution. Compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 3 - \lambda & 1 \\ 1 & 3 - \lambda \end{vmatrix} = (3 - \lambda)^2 - 1.$$

Expanding,

$$(3 - \lambda)^2 - 1 = \lambda^2 - 6\lambda + 8,$$

and we arrive at the characteristic polynomial

$$\lambda^2 - 6\lambda + 8 = 0$$

We can also use equation (2). We calculate the trace and determinant as

$$\operatorname{tr}(A) = 3 + 3 = 6, \quad \det(A) = 3 \cdot 3 - 1 \cdot 1 = 8.$$

Thus, the characteristic equation is

$$\lambda^2 - 6\lambda + 8 = 0,$$

which matches our previous computation. Factoring the characteristic polynomial yields

$$(\lambda - 2)(\lambda - 4) = 0.$$

Therefore, the eigenvalues are

$$\lambda = 2, \quad 4.$$

These are two distinct real eigenvalues. ☐

Example 2.7. Find the eigenvalues of

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

Solution. Compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 0 & 2-\lambda \end{vmatrix} = (2-\lambda)^2.$$

Thus,

$$(2-\lambda)^2 = 0,$$

so the only eigenvalue is

$$\lambda = 2,$$

with multiplicity 2. This is an example of a repeated (or defective) eigenvalue. ☐

Example 2.8. Find the eigenvalues of

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Solution. Compute the characteristic polynomial:

$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 + 1.$$

Thus,

$$\lambda^2 + 1 = 0,$$

so the eigenvalues are

$$\lambda = \pm i.$$

These are complex conjugate eigenvalues. ☐

2.3.2 The case of a 3×3 matrix

For a 3×3 matrix, the characteristic polynomial is cubic. In principle, one can always find the eigenvalues by computing

$$\det(A - \lambda I) = 0,$$

but it is useful to know the general form of this polynomial in advance. This gives a shortcut for checking calculations and organizing the computation. Let

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}.$$

Then

$$A - \lambda I = \begin{pmatrix} a-\lambda & b & c \\ d & e-\lambda & f \\ g & h & k-\lambda \end{pmatrix}.$$

The characteristic polynomial is

$$p(\lambda) = \det(A - \lambda I).$$

Since this determinant is taken from a 3×3 matrix whose diagonal entries each contain a term linear in λ, the result is a polynomial of degree 3. In fact, it has the general form

$$p(\lambda) = -\lambda^3 + (\operatorname{tr} A)\lambda^2 - \sigma_2(A)\lambda + \det(A),$$

where $\sigma_2(A)$ is the sum of the three 2×2 principal minors of A. Equivalently, many texts write the characteristic equation as

$$\lambda^3 - (\operatorname{tr} A)\lambda^2 + \sigma_2(A)\lambda - \det(A) = 0.$$

The coefficients of the characteristic polynomial have useful meanings:

- The coefficient of λ^2 is the trace:

$$\operatorname{tr}(A) = a + e + k.$$

- The coefficient of λ is the sum of the principal 2×2 minors:

$$\sigma_2(A) = \begin{vmatrix} a & b \\ d & e \end{vmatrix} + \begin{vmatrix} a & c \\ g & k \end{vmatrix} + \begin{vmatrix} e & f \\ h & k \end{vmatrix}.$$

- The constant term is the determinant:

$$\det(A).$$

So the characteristic polynomial can be written as

$$p(\lambda) = -\lambda^3 + (a+e+k)\lambda^2 - \sigma_2(A)\lambda + \det(A).$$

We have the following useful formula for the 3×3 case:

$$\boxed{p(\lambda) = \lambda^3 - (\operatorname{tr} A)\lambda^2 + \sigma_2(A)\lambda - \det(A)} \tag{3}$$

where $\sigma_2(A)$ is the sum of the three principal 2×2 minors. This is often much faster than expanding $\det(A - \lambda I)$ from scratch, especially when the matrix has a simple structure. This is especially helpful when the matrix has integer entries, since the Rational Root Theorem often makes it possible to guess one root and then factor the polynomial.

Example 2.9. Let

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 2 \end{pmatrix}.$$

Solution. Since A is triangular, the eigenvalues are the diagonal entries:

$$\lambda = 1,\ 3,\ 2.$$

We can also verify this with the shortcut formula. The trace is

$$\operatorname{tr}(A) = 1 + 3 + 2 = 6,$$

and the determinant is

$$\det(A) = 1 \cdot 3 \cdot 2 = 6.$$

The principal 2×2 minors are

$$\begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3, \qquad \begin{vmatrix} 1 & 0 \\ 4 & 2 \end{vmatrix} = 2, \qquad \begin{vmatrix} 3 & 0 \\ 5 & 2 \end{vmatrix} = 6.$$

So

$$\sigma_2(A) = 3 + 2 + 6 = 11.$$

Thus the characteristic polynomial is

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6,$$

which factors as

$$(\lambda - 1)(\lambda - 2)(\lambda - 3).$$

So the eigenvalues are indeed $1, 2, 3$. □

Example 2.10. Find the eigenvalues of

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 6 & -11 & 6 \end{pmatrix}$$

This matrix is *not* triangular, but the characteristic polynomial still factors nicely.
Step 1: Compute the trace.

$$\operatorname{tr}(A) = 0 + 0 + 6 = 6.$$

Step 2: Compute the sum of the principal 2×2 minors. The three principal 2×2 minors are

$$\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 0 \\ 6 & 6 \end{vmatrix} = 0, \qquad \begin{vmatrix} 0 & 1 \\ -11 & 6 \end{vmatrix} = 11.$$

So

$$\sigma_2(A) = 0 + 0 + 11 = 11.$$

Step 3: Compute the determinant. Expanding along the first row,

$$\det(A) = 0 - 1 \begin{vmatrix} 0 & 1 \\ 6 & 6 \end{vmatrix} + 0.$$

Since

$$\begin{vmatrix} 0 & 1 \\ 6 & 6 \end{vmatrix} = -6,$$

we get

$$\det(A) = 6.$$

Step 4: Write the characteristic polynomial. Using equation 3

$$\lambda^3 - (\operatorname{tr} A)\lambda^2 + \sigma_2(A)\lambda - \det(A) = 0,$$

we obtain

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = 0.$$

Trying $\lambda = 1$ gives

$$1 - 6 + 11 - 6 = 0,$$

so $(\lambda - 1)$ is a factor. Dividing, we get

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda^2 - 5\lambda + 6).$$

Then

$$\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3).$$

Therefore,

$$\lambda^3 - 6\lambda^2 + 11\lambda - 6 = (\lambda - 1)(\lambda - 2)(\lambda - 3).$$

We arrive at the following three eigenvalues:

$$\lambda = 1,\ 2,\ 3.$$

2.4 Useful Shortcuts

Some matrices have special forms that make eigenvalues easy to identify.

Diagonal matrices. If

$$A = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{pmatrix},$$

then the eigenvalues are simply the diagonal entries $d_1, \ldots, d_n$.

Triangular matrices. If A is upper triangular or lower triangular, then its eigenvalues are also the diagonal entries. This is because the determinant of $A - \lambda I$ is the product of the diagonal terms.

3 Eigenvectors

In the previous section, we learned how to compute eigenvalues. We now return to the original motivation: understanding vectors that behave in a simple way under a linear transformation.

Eigenvectors are the vectors that correspond to eigenvalues, and together they provide a powerful way to understand and simplify matrix operations.

Definition 3.1. Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. A nonzero vector $\vec{\alpha}$ is called an **eigenvector** corresponding to λ if

$$A\vec{\alpha} = \lambda\vec{\alpha}.$$

This equation says that applying A to $\vec{\alpha}$ simply scales the vector by λ—it does not change its direction (except possibly reversing it if $\lambda < 0$). Eigenvectors are directions that are preserved by the transformation and tell us how much scaling occurs in those directions.

3.1 How to find eigenvectors

Once an eigenvalue λ is known, we find eigenvectors by solving

$$\boxed{(A - \lambda I)\vec{\alpha} = 0.}$$

This is a homogeneous system of linear equations. The solutions form a vector space called the **eigenspace** corresponding to λ. The general procedure is as follows:

1. Compute $A - \lambda I$,
2. Solve the system $(A - \lambda I)\vec{\alpha} = 0$,
3. Express the solution in parametric form,
4. Any nonzero solution is an eigenvector.

Example 3.2. Find the eigenvectors of

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}.$$

Solution. We already found the eigenvalues:

$$\lambda_1 = 2, \quad \lambda_2 = 4.$$

Case 1: $\lambda = 2$: We compute the eigenvector associated to the eigenvalue $\lambda_1 = 2$ by solving the following system

$$(A - 2I)\vec{\alpha}_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \vec{\alpha} = \vec{0}.$$

which yields

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

This gives the equation $\alpha_1 + \alpha_2 = 0$, so $\alpha_2 = -\alpha_1$. Thus,

$$\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ -\alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So the eigenvectors are all multiples of

$$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Case 2: $\lambda = 4$ We compute the eigenvector associated to the eigenvalue $\lambda_2 = 4$ by solving the following system

$$(A - 4I)\vec{\alpha}_2 = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \vec{\alpha}_2$$

which yields

$$\alpha_1 - \alpha_2 = 0 \quad \Rightarrow \quad \alpha_1 = \alpha_2.$$

Thus we have the eigenvector of

$$\vec{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_1 \end{pmatrix} = \alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So the eigenvectors are all multiples of

$$\vec{\alpha}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

□

Example 3.3. Find the eigenvectors of

$$A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

Solution. Since A is upper triangular, the eigenvalues are the diagonal entries:

$$\lambda = 2,\ 3,\ 4.$$

We now find an eigenvector for each eigenvalue.
Eigenvectors for $\lambda = 2$. Substituting , we solve the system

$$(A - 2I)\vec{\alpha}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \vec{\alpha}_1 = \vec{0}.$$

This gives the system

$$\alpha_2 = 0, \qquad \alpha_3 = 0.$$

So α_1 is free, and the eigenvectors are all multiples of

$$\vec{\alpha}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Eigenvectors for $\lambda = 3$. Substituting $\lambda = 3$, we solve the system

$$(A - 3I)\vec{\alpha}_2 = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \vec{\alpha}_2 = \vec{0}.$$

Solving the above system gives

$$-\alpha_1 + \alpha_2 = 0, \qquad \alpha_3 = 0.$$

Thus $\alpha_1 = \alpha_2$, $\alpha_3 = 0$, and the eigenvectors are all multiples of

$$\vec{\alpha_2} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}.$$

Eigenvectors for $\lambda = 4$. Substituting $\lambda = 4$, we solve the system

$$(A - 4I)\vec{\alpha}_3 = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \vec{\alpha}_3 = \vec{0}.$$

Solving the above system gives

$$-\alpha_2 = 0 \Rightarrow \alpha_2 = 0,$$

and then

$$-2\alpha_1 + \alpha_2 = 0 \Rightarrow \alpha_1 = 0.$$

So α_3 is free, and the eigenvectors are all multiples of

$$\vec{\alpha}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

The eigenvectors are

$$\lambda = 2: \ \text{span}\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\}, \qquad \lambda = 3: \ \text{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}, \qquad \lambda = 4: \ \text{span}\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}.$$

□

3.2 Repeated eigenvalues

So far, we have focused on the case where all eigenvalues are distinct. In that situation, each eigenvalue produces one linearly independent eigenvector, and the structure of the solution is straightforward.

However, new phenomena arise when an eigenvalue is **repeated**. In this case, it is important to distinguish between two different concepts: **algebraic multiplicity** and **geometric multiplicity**.

Definition 3.4. Let λ be an eigenvalue of a matrix A.

- The **algebraic multiplicity** of λ is its multiplicity as a root of the characteristic polynomial.
- The **geometric multiplicity** of λ is the dimension of the eigenspace

$$\{\mathbf{v} : (A - \lambda I)\mathbf{v} = 0\}.$$

We have the following inequality that describes the relationship between the algebraic and geometric multiplicity:

$$1 \leq \text{geometric multiplicity} \leq \text{algebraic multiplicity}.$$

Thus, a repeated eigenvalue does not necessarily produce multiple independent eigenvectors. If the geometric multiplicity equals the algebraic multiplicity, then there are enough independent eigenvectors to match the multiplicity.

Example 3.5. Consider

$$A = \begin{pmatrix}2 & 0\\0 & 2\end{pmatrix}.$$

The characteristic equation is

$$(2 - \lambda)^2 = 0,$$

so $\lambda = 2$ has algebraic multiplicity 2. Compute

$$A - 2I = \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}.$$

Thus every vector is an eigenvector, so the eigenspace has dimension 2. Therefore, the geometric multiplicity is also 2.

This matrix has two independent eigenvectors and behaves like a diagonal matrix.

Definition 3.6. If the geometric multiplicity is *less* than the algebraic multiplicity, then the matrix does not have enough independent eigenvectors. Such a matrix is called **defective**.

Example 3.7. Consider

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}.$$

The characteristic equation is

$$(2 - \lambda)^2 = 0,$$

so $\lambda = 2$ has algebraic multiplicity 2. Compute

$$A - 2I = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Solving

$$(A - 2I)\mathbf{v} = 0$$

gives $y = 0$, so

$$\mathbf{v} = \begin{pmatrix} x \\ 0 \end{pmatrix}.$$

Thus, there is only one linearly independent eigenvector. The geometric multiplicity is 1. This matrix is defective.

Proposition 3.8. *A defective matrix cannot be diagonalized.*

Repeated Complex Eigenvalues

Repeated eigenvalues may also be complex. Since matrices with real entries have complex eigenvalues in conjugate pairs, repeated complex eigenvalues occur as repeated conjugate pairs.

Example 3.9. Consider a matrix with characteristic polynomial

$$(\lambda^2 + 4)^2 = 0.$$

Then the eigenvalues are

$$\lambda = \pm 2i,$$

each with multiplicity 2.

In this case:

- The algebraic multiplicity of each complex eigenvalue is 2,
- The corresponding eigenspaces are computed over $\mathbb{C}$,
- When working over $\mathbb{R}$, these lead to solutions involving sine and cosine functions (as seen in differential equations).

Just as in the real case, it is possible for repeated complex eigenvalues to be defective, meaning there are not enough independent eigenvectors.

4 Geometric meaning of eigenvalues and eigenvectors

4.1 Invariant directions

A linear transformation A typically changes both the length and direction of a vector. However, there are special vectors $\vec{\alpha}$ for which the transformation does not change direction. Instead, the vector is simply scaled:

$$A\vec{\alpha} = \lambda\vec{\alpha}.$$

Geometrically, this means that $\vec{\alpha}$ lies along a direction that is preserved by the transformation. Any vector on the line spanned by $\vec{\alpha}$ remains on that same line after applying A. These directions are called invariant directions, and they capture the simplest behavior of the transformation.

4.2 Scaling along eigenvector directions

The scalar λ describes how the transformation acts along the direction $\vec{\alpha}$.

- If $|\lambda| > 1$, vectors are stretched.
- If $0 < |\lambda| < 1$, vectors are compressed.
- If $\lambda < 0$, the direction is reversed.
- If $\lambda = 1$, vectors are unchanged.

Thus, while most vectors are distorted in complicated ways, eigenvectors $\vec{\alpha}$ experience only simple scaling. This is why they are so useful: they reveal directions in which the transformation behaves in the most understandable way.

4.3 Connection to diagonalization

When a matrix has enough independent eigenvectors $\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3$, these vectors can be used as a new coordinate system. In this new basis, the transformation A no longer mixes directions; instead, it acts by scaling each coordinate independently:

$$A\vec{\alpha}_i = \lambda_i\vec{\alpha}_i.$$

Geometrically, this means that although the transformation may look complicated in the standard coordinate system, there exists a special set of directions in which the action of A becomes completely simple. Diagonalization is precisely the process of finding this coordinate system.

5 Diagonalization

5.1 Introduction

Diagonalization is a fundamental concept in linear algebra that simplifies matrix computations by transforming a matrix into a diagonal form. A diagonal matrix is easier to work with, especially when computing powers of matrices, exponentials, and solving systems of differential equations.

Definition 5.1. A square matrix $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) is said to be **diagonalizable** if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

Equivalently,

$$D = P^{-1}AP.$$

Theorem 5.2 (Diagonalization Theorem)**.** *An $n \times n$ matrix A is diagonalizable if and only if it has n linearly independent eigenvectors.*

Sketch of Proof. If A has n linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, form the matrix

$$P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n].$$

Then

$$AP = PD,$$

where D is diagonal with eigenvalues $\lambda_1, \dots, \lambda_n$ on the diagonal. Multiplying by P^{-1} gives

$$A = PDP^{-1}.$$

□

Theorem 5.3. *A matrix is diagonalizable if and only if, for each eigenvalue, the geometric multiplicity equals the algebraic multiplicity.*

5.2 Finding the diagonalization

Given a square matrix A, diagonalization consists of finding an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

The columns of P will be **eigenvectors** of A, and the diagonal entries of D will be the **corresponding eigenvalues.**

To diagonalize a matrix A, follow these steps:

1. Compute the characteristic polynomial:

$$\det(A - \lambda I) = 0.$$

2. Solve for eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$.
3. For each eigenvalue λ_i, find eigenvectors by solving:
$$(A - \lambda_i I)\mathbf{v} = 0.$$
4. Check that you have n linearly independent eigenvectors.
    - If yes: the matrix is diagonalizable.
    - If not: the matrix is not diagonalizable.
5. Form the matrix P using eigenvectors as columns:
$$P = [\mathbf{v}_1 \ \mathbf{v}_2 \ \cdots \ \mathbf{v}_n].$$
6. Form the diagonal matrix D:
$$D = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n).$$
7. Verify:
$$P^{-1}AP = D.$$

Example 5.4. Diagonalize the matrix
$$A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}.$$

Solution. We first find the eigenvalues by calculating the characteristic polynomial:
$$\det(A - \lambda I) = \lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3) = 0.$$
Hence, the eigenvalues are:
$$\lambda_1 = 3, \quad \lambda_2 = 2.$$
The next step is to find the corresponding eigenvectors. For $\lambda = 3$:
$$(A - 3I)\vec{\alpha} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \vec{\alpha} = 0.$$
This gives $\alpha_2 = 0$, so
$$\vec{\alpha}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$
Next, for $\lambda = 2$:
$$(A - 2I)\vec{\alpha} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \vec{\alpha} = 0.$$
This gives $\alpha_1 = -\alpha_2$, so
$$\vec{\alpha}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$
It follows that we can write P and D as the following:
$$P = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}.$$
□

Example 5.5. Diagonalize the matrix

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

Solution. First, we find the eigenvalues of A as:

$$\det(A - \lambda I) = (4 - \lambda)^2.$$

So $\lambda = 4$ with multiplicity 2. Next, we see if we can find two linearly independent eigenvectors corresponding to the sole eigenvalue. We solve the following system

$$(A - 4I)\vec{\alpha} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{\alpha} = \vec{0}.$$

We have that

$$\alpha_2 = 0.$$

Thus eigenvectors are:

$$\vec{\alpha} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

There is only *one* independent eigenvector. Therefore, A is *not* diagonalizable. □

Example 5.6. Diagonalize the matrix

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 1 \\ 0 & 1 & 3 \end{bmatrix}.$$

Solution. We first find the eigenvalues of A as

$$\det(A - \lambda I) = \det \begin{bmatrix} 2-\lambda & 0 & 0 \\ 0 & 3-\lambda & 1 \\ 0 & 1 & 3-\lambda \end{bmatrix} = (2-\lambda) \begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix}.$$

It follows that

$$\det(A - \lambda I) = (2 - \lambda)\left[(3-\lambda)^2 - 1\right],$$

which is factorable as

$$(3-\lambda)^2 - 1 = (3 - \lambda - 1)(3 - \lambda + 1) = (2 - \lambda)(4 - \lambda).$$

So, the characteristic polynomial is:

$$(2-\lambda)^2(4-\lambda).$$

The eigenvalues are:

$$\lambda = 2 \quad \text{(algebraic multiplicity 2)}, \qquad \lambda = 4 \quad \text{(algebraic multiplicity 1)}.$$

Next, we find the associated eigenvectors to each eigenvalue. First, we have

$$(A-4I)\vec{\alpha}_1 = 0.$$

$$(A-4I)\vec{\alpha}_1 = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \vec{\alpha}_1 = \vec{0}.$$

This gives the system:

$$-2x = 0, \quad -y+z = 0, \quad y-z = 0$$

and hence

$$x = 0, \quad y = z.$$

Thus, the eigenvector is

$$\vec{\alpha}_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

In order to diagonalize A, we need to find two *linearly independent* eigenvectors corresponding to the eigenvalue $\lambda = 2$. Accordingly, we solve the following system :

$$(A-2I)\vec{\alpha}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \vec{\alpha}_2 = \vec{0}$$

This gives:

$$y + z = 0.$$

So:

$$z = -y, \quad x \text{ is free.}$$

Thus the general eigenvector is:

$$\vec{\alpha}_2 = \begin{bmatrix} x \\ y \\ -y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

So we obtain two independent eigenvectors:

$$\vec{\alpha}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{\alpha}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

We take eigenvectors as columns:

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}.$$

Matching the order of columns in P, we take:

$$D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

Since we found three linearly independent eigenvectors, the matrix is diagonalizable, and we have:

$$A = PDP^{-1}.$$

This matrix stretches space along three independent directions:

- One direction (spanned by $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$) is stretched by a factor of 4.
- Two independent directions (spanned by $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$) are stretched by a factor of 2.

Thus, even though the matrix is not diagonal in the standard basis, it behaves like a diagonal matrix in the basis of its eigenvectors. □

5.3 Matrix Powers

If $A = PDP^{-1}$, then

$$A^k = PD^kP^{-1}.$$

Since D is diagonal, we can compute powers of D in the following way:

$$D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}.$$

Example 5.7. Calculate A^k is the matric A is given by

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}.$$

Solution. From previous computations, the eigenvalues are $\lambda_1 = 5$, $\lambda_2 = 2$, with eigenvectors:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

Thus,

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}, \quad D = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}.$$

$$D^k = \begin{bmatrix} 5^k & 0 \\ 0 & 2^k \end{bmatrix}.$$

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

$$A^k = PD^kP^{-1}.$$

First compute:

$$PD^k = \begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 5^k & 0 \\ 0 & 2^k \end{bmatrix} = \begin{bmatrix} 5^k & -2^k \\ 5^k & 2 \cdot 2^k \end{bmatrix}.$$

Now multiply by P^{-1}:

$$A^k = \frac{1}{3}\begin{bmatrix} 5^k & -2^k \\ 5^k & 2 \cdot 2^k \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}.$$

Carrying out the multiplication:

$$A^k = \frac{1}{3}\begin{bmatrix} 2 \cdot 5^k + 2^k & 5^k - 2^k \\ 2 \cdot 5^k - 2 \cdot 2^k & 5^k + 2 \cdot 2^k \end{bmatrix}.$$

□

6 Jordan Canonical Form

6.1 What is the Jordan form?

Not every matrix is diagonalizable. That is, some matrices do not have enough linearly independent eigenvectors to form a basis. From our language so far, that means that some eigenvalues have a difference between their algebraic and geometric multiplicities. In such cases, diagonalization fails, and we need a more general way to simplify matrices. The **Jordan canonical form** provides the next best thing.

Definition 6.1. A matrix J is called a **Jordan matrix** if it has the block-diagonal form

$$J = \begin{bmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{bmatrix},$$

where each block J_i is a **Jordan block** of the form

$$J_i = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ 0 & 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}.$$

Instead of fully diagonalizing a matrix, we bring it as close as possible to diagonal form. The only extra complication is the appearance of 1's just above the diagonal. The Jordan canonical form allows for the following:

- It generalizes diagonalization: diagonal matrices are special cases of Jordan form.
- It reveals the full structure of a linear transformation.
- It allows computation of matrix functions (e.g., A^k, e^A).
- It explains what happens when there are not enough eigenvectors.

Thinking geometrically, diagonalizable matrices act as pure scaling along independent directions.Jordan blocks represent **almost scaling**, but with a small "drift" in one direction. This drift corresponds to generalized eigenvectors.

6.2 Computing the Jordan form

To compute the Jordan form, we must go beyond ordinary eigenvectors.
Step 1: Find Eigenvalues First, we find the eigenvalues by solving the following polynomial:

$$\det(A - \lambda I) = 0$$

to obtain eigenvalues and their **algebraic multiplicities**.
Step 2: Find Eigenvectors Next, we find the eigenvectors by solving the following system:

$$(A - \lambda I)\mathbf{v} = 0.$$

Count the number of independent eigenvectors, which is the geometric multiplicity. We have the following two possibilities:

- If geometric multiplicity = algebraic multiplicity for all eigenvalues: diagonalizable.
- Otherwise: Jordan blocks are required.

Step 4: Find Generalized Eigenvectors If there are not enough eigenvectors, then we solve the system solve:

$$(A - \lambda I)^k\mathbf{v} = 0$$

for increasing powers k.

Definition 6.2. A **generalized eigenvector** satisfies:

$$(A - \lambda I)^k\mathbf{v} = 0 \quad \text{but} \quad (A - \lambda I)^{k-1}\mathbf{v} \neq 0.$$

Step 5: Build Jordan Chains A Jordan chain has the form:

$$(A - \lambda I)\mathbf{v}_2 = \mathbf{v}_1, \quad (A - \lambda I)\mathbf{v}_1 = 0.$$

where

- $\mathbf{v}_1$ is an eigenvector.
- $\mathbf{v}_2$ is a generalized eigenvector.

Each chain corresponds to one Jordan block.
Step 6: Construct P and J

- Columns of P: eigenvectors and generalized eigenvectors (ordered by chains).
- J: Jordan blocks corresponding to those chains.

Example 6.3. Find the Jordan form of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Solution. We first calculate the eigenvalues of A as

$$(1 - \lambda)^2 = 0 \Rightarrow \lambda = 1.$$

Hence, there is one eigenvalues $\lambda = 1$ whose algebraic multiplicity is 2. In order for A to be diagonalizable, there have to be *two* linearly independent eigenvectors. We calculate the eigenvectors by solving the following system:

$$(A - I)\mathbf{v} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{v} = 0.$$

This gives only one eigenvector:

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Hence, the geometric multiplicity is one and A is *not* diagonalizable. Therefore, the Jordan block is

$$J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

Note that because the geometric multiplicity does not equal the algebraic multiplicity, there is a 1 in the entry above the second diagonal 1.

Next, in order to find P, we have to find a generalized eigenvector by solving the following system for $\mathbf{v}_2$:

$$(A - I)\mathbf{v}_2 = \mathbf{v}_1.$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

This gives $y = 1$, so:

$$\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Therefore, we have

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad J = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

□

Example 6.4. Find the Jordan form of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

Solution. First, we find the eigenvalues of A to solve the following equation:

$$(2-\lambda)^3 = 0 \Rightarrow \lambda = 2.$$

To find the eigenvector, we solve the following system:

$$(A-2I)\vec{\alpha} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \vec{\alpha} = \vec{0}.$$

Solving the system yields only one eigenvector:

$$\vec{\alpha} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, the geometric multiplicity of $\lambda = 2$ is 1. Therefore, the Jordan form is given by

$$J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}.$$

To find the two missing columns in P, we construct the following chain:

$$(A-2I)\vec{\alpha}_2 = \vec{\alpha}_1, \quad (A-2I)\vec{\alpha}_3 = \vec{\alpha}_2.$$

This produces a Jordan chain of length 3. Thus, □

Example 6.5. Find the Jordan form of the following matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & -1 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Solution. Since A is upper triangular, its eigenvalues are the diagonal entries. Thus the characteristic polynomial is

$$\det(A-\lambda I) = (2-\lambda)^2(5-\lambda)^2.$$

So the eigenvalues are

$$\lambda_1 = 2 \quad \text{(algebraic multiplicity 2)}, \qquad \lambda_2 = 5 \quad \text{(algebraic multiplicity 2)}.$$

Next, we find the eigenvector associated to the eigenvalue $\lambda_1 = 2$ by solving the system

$$(A-2I)\vec{\alpha}_1 = \begin{bmatrix} 0 & 1 & 3 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} \vec{\alpha}_1 = \vec{0}.$$

We have that

$$\mathbf{v} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}.$$

and hence

$$(A-2I)\vec{\alpha}_1 = \begin{bmatrix} \alpha_2 + 3\alpha_3 - \alpha_4 \\ 3\alpha_4 \\ 3\alpha_3 \\ 3\alpha_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the last three equations, we get

$$\alpha_4 = 0, \qquad \alpha_3 = 0.$$

Then the first equation gives

$$\alpha_2 = 0.$$

So α_1 is free, and the eigenspace is

$$E_2 = \left\{ \begin{bmatrix} \alpha_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} : \alpha_1 \in \mathbb{R} \right\}.$$

Thus, there is only one eigenvector:

$$\vec{\alpha}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So $\lambda = 2$ has geometric multiplicity 1, and we need one generalized eigenvector. We look for $\vec{\alpha}_2$ such that

$$(A-2I)\vec{\alpha}_2 = \vec{\alpha}_1.$$

Similar to above, we have that

$$\vec{\alpha}_2 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}.$$

The system then becomes

$$(A-2I)\vec{\alpha}_2 = \begin{bmatrix} \alpha_2 + 3\alpha_3 - \alpha_4 \\ 3\alpha_4 \\ 3\alpha_3 \\ 3\alpha_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

So

$$3\alpha_4 = 0 \Rightarrow \alpha_4 = 0, \qquad 3\alpha_3 = 0 \Rightarrow \alpha_3 = 0,$$

and then

$$\alpha_2 = 1.$$

A convenient choice is

$$\vec{\alpha}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}.$$

So, the Jordan chain for $\lambda = 2$ is

$$\vec{\alpha}_1, \vec{\alpha}_2.$$

Next, we find the eigenvectors to the associated eigenvalue $\lambda_2 = 5$ by solving the following system

$$(A-5I)\vec{\alpha}_3 = \begin{bmatrix} -3 & 1 & 3 & -1 \\ 0 & -3 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vec{\alpha}_2 = 0.$$

Let

$$\vec{\alpha}_3 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix}.$$

Then

$$(A-5I)\vec{\alpha}_3 = \begin{bmatrix} -3\alpha_1 + \alpha_2 + 3\alpha_3 - \alpha_4 \\ -3\alpha_2 + 3\alpha_4 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

From the second equation,

$$-3\alpha_2 + 3\alpha_4 = 0 \quad \Rightarrow \quad \alpha_2 = \alpha_4.$$

and substituting into the first equation, we have

$$3\alpha_1 + 3\alpha_3 = 0,$$

so

$$\alpha_1 = \alpha_3.$$

Thus, the eigenvectors for $\lambda = 5$ have the form

$$\vec{\alpha}_{3,4} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} = \alpha_3 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + \alpha_4 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

So we get two independent eigenvectors:

$$\vec{\alpha}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \vec{\alpha}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore $\lambda = 5$ has geometric multiplicity 2, so it contributes two 1×1 Jordan blocks. It follows that we have

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The Jordan form is

$$J = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Indeed,

$$P^{-1}AP = J.$$

□