

Exam 2 Practice Problems Solutions

1. Use the integral comparison test to show if the following integrals converge or diverge.

(a) $\int_1^{\infty} \frac{1}{x^{2026} + 2026} dx$

Solution. We can bound the integrand in the following way:

$$x^{2026} + 2026 > x^{2026} \Rightarrow \frac{1}{x^{2026} + 2026} < \frac{1}{x^{2026}} \quad \text{for } x \geq 1$$

Hence, we can bound $\int_0^{\infty} \frac{1}{x^{2026} + 2026} dx$ from below by $\int_1^{\infty} \frac{1}{x^{2026}} dx$ which converges from the p -integral test. The Integral Comparison Test therefore yields that $\int_0^{\infty} \frac{1}{x^{2026} + 2026} dx$ converges. $\square$

(b) $\int_1^{\infty} \frac{1}{x + e^{2x}} dx$

Solution. For all $x \geq 1$, we have

$$x + e^{2x} > e^{2x} \Rightarrow 0 < \frac{1}{x + e^{2x}} < \frac{1}{e^{2x}} = e^{-2x}.$$

Now consider the comparison integral which we can evaluate in the following way:

$$\begin{aligned} \int_1^{\infty} e^{-2x} dx &= \lim_{b \rightarrow \infty} \left[-\frac{1}{2} e^{-2x} \right]_1^b \\ &= \lim_{b \rightarrow \infty} \left(-\frac{1}{2} e^{-2b} + \frac{1}{2} e^{-2} \right) \\ &= \frac{1}{2e^2} < \infty. \end{aligned}$$

Since $0 < \frac{1}{x + e^{2x}} < e^{-2x}$ and $\int_1^{\infty} e^{-2x} dx$ converges, the Direct Comparison Test implies that $\int_1^{\infty} \frac{1}{x + e^{2x}} dx$ converges. $\square$

2. Determine whether the following series converge. State what tests you are using and verify the hypotheses.

(a) $\sum_{n=0}^{\infty} \frac{(-1)^n}{4n+1}$

Solution. Set $b_n = \frac{1}{4n+1}$. Clearly, $b_n > 0$ for all $n \geq 0$. Moreover,

$$b_{n+1} = \frac{1}{4n+5} < \frac{1}{4n+1} = b_n,$$

so (b_n) is decreasing. Finally,

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} \frac{1}{4n+1} = 0.$$

Thus, all hypotheses of the Alternating Series Test are satisfied, and the series converges. To determine whether convergence is absolute, observe that

$$\lim_{n \rightarrow \infty} \frac{\frac{1}{4n+1}}{\frac{1}{n}} = \lim_{n \rightarrow \infty} \frac{n}{4n+1} = \frac{1}{4}.$$

Since this limit is finite and positive and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, the Limit Comparison Test implies that the series of absolute values diverges. Therefore, the series converges conditionally. $\square$

(b) $\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{n^4+1}}$

Solution. All terms are positive. We apply the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. Indeed,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{\frac{n+2}{\sqrt{n^4+1}}}{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \frac{n(n+2)}{\sqrt{n^4+1}} \\ &= \lim_{n \rightarrow \infty} \frac{1 + \frac{2}{n}}{\sqrt{1 + \frac{1}{n^4}}} \\ &= 1. \end{aligned}$$

Since the limit is finite and strictly positive, the Limit Comparison Test yields that both series have the same convergence behavior. The harmonic series diverges by the p -series test with $p = 1$. Hence, the series diverges. $\square$

(c) $\sum_{n=3}^{\infty} \frac{\ln(n)}{n^3}$

(d) $\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

Solution. Set $a_n = \frac{n^n}{(n!)^2} > 0$. We apply the Ratio Test and compute

$$\begin{aligned} \frac{a_{n+1}}{a_n} &= \frac{(n+1)^{n+1} (n!)^2}{((n+1)!)^2 n^n} \\ &= \frac{(n+1)^{n-1}}{n^n} \\ &= \frac{1}{n+1} \left(1 + \frac{1}{n}\right)^n. \end{aligned}$$

Hence,

$$L = \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \lim_{n \rightarrow \infty} \frac{(1 + 1/n)^n}{n + 1} = 0,$$

because $(1 + 1/n)^n \rightarrow e$. Since $L = 0 < 1$, the Ratio Test implies that the series converges. $\square$

$$(e) \sum_{n=3}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$$

Solution. Let $a_n = \ln(n)^{\ln(n)}$. Using properties of logs and exponentials, we have $a_n = \ln(n)^{\ln(n)} = e^{\ln(n)\ln(\ln(n))} = n^{\ln(\ln(n))} > n^2$. Hence

$$\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and the Direct Comparison Test yields that $\sum_{n=1}^{\infty} a_n$ converges as $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges by the p-series test. $\square$

$$(f) \sum_{n=1}^{\infty} \ln \left(\frac{n^5}{5n^5 + 4n^3 + 1} \right)$$

Solution. Set $a_n = \ln \left(\frac{n^5}{5n^5 + 4n^3 + 1} \right)$. A necessary condition for a series to converge is that its terms approach zero. However,

$$\begin{aligned} \lim_{n \rightarrow \infty} a_n &= \ln \left(\lim_{n \rightarrow \infty} \frac{n^5}{5n^5 + 4n^3 + 1} \right) \\ &= \ln \left(\frac{1}{5} \right) \neq 0. \end{aligned}$$

Here we used the continuity of the log on $(0, \infty)$. Since the terms do not approach zero, the Divergence Test gives that the series diverges. $\square$

$$(g) \sum_{n=1}^{\infty} \left(\frac{n}{n^4 - 3n^2 + 3^n} \right)^{2n}$$

Solution. Set $a_n = \left(\frac{n}{n^4 - 3n^2 + 3^n} \right)^{2n}$. For $n \geq 1$, the denominator is positive, so $a_n > 0$. We apply the Root Test by calculating

$$\begin{aligned} \sqrt[n]{a_n} &= \left(\frac{n}{n^4 - 3n^2 + 3^n} \right)^2 \\ &= \frac{1}{(n^3 - 3n + 3)^2}. \end{aligned}$$

Therefore,

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = 0.$$

Since $L = 0 < 1$, the Root Test yields that the series converges. $\square$

$$(h) \sum_{n=2}^{\infty} \frac{\ln(\ln(n))}{n \ln(n)}$$

Solution. We apply the Integral Test to $f(x) = \frac{\ln(\ln x)}{x \ln x}$. For $x \geq e^e$, the function is continuous and positive. Differentiating gives

$$f'(x) = \frac{1 - (\ln x + 1) \ln(\ln x)}{x^2 (\ln x)^2}.$$

For $x \geq e^e$, we have $\ln(\ln x) \geq 1$, so

$$1 - (\ln x + 1) \ln(\ln x) \leq -\ln x < 0.$$

Thus f is decreasing on $[e^e, \infty)$. All hypotheses of the Integral Test are therefore satisfied on this interval. Using the substitution $u = \ln(\ln x)$, so that $du = dx/(x \ln x)$, we obtain

$$\begin{aligned} \int_{e^e}^{\infty} \frac{\ln(\ln x)}{x \ln x} dx &= \int_1^{\infty} u du \\ &= \lim_{b \rightarrow \infty} \left[\frac{u^2}{2} \right]_1^b \rightarrow \infty \end{aligned}$$

As the corresponding integral does not converge, the Integral Test provides that the series does not converge either. $\square$

$$(i) \sum_{n=2}^{\infty} \frac{\cos(n)}{n^5 + n^3 + 2}$$

Solution. We show that the series converges absolutely. Since $|\cos n| \leq 1$ and the denominator is positive, we have

$$0 \leq \left| \frac{\cos n}{n^5 + n^3 + 2} \right| \leq \frac{1}{n^5}.$$

The series $\sum_{n=2}^{\infty} \frac{1}{n^5}$ converges by the p -series test, since $p = 5 > 1$. Thus, by the Direct Comparison Test, $\sum_{n=2}^{\infty} \left| \frac{\cos n}{n^5 + n^3 + 2} \right|$ converges. Therefore, the series converges absolutely. $\square$

$$(j) \sum_{n=1}^{\infty} (-1)^{n-1} (\ln(n+1) - \ln(n))$$

Solution. Set $b_n = \ln(n+1) - \ln n = \ln\left(1 + \frac{1}{n}\right)$. Since $1 + 1/n > 1$, we have $b_n > 0$. Moreover, $1/(n+1) < 1/n$, and the logarithm is strictly increasing. Hence,

$$b_{n+1} = \ln\left(1 + \frac{1}{n+1}\right) < \ln\left(1 + \frac{1}{n}\right) = b_n.$$

Finally, by continuity of the logarithm,

$$\lim_{n \rightarrow \infty} b_n = \ln(1) = 0.$$

All hypotheses of the Alternating Series Test hold, so the series converges. To test absolute convergence, observe that

$$\begin{aligned} \sum_{n=1}^N |(-1)^{n-1} b_n| &= \sum_{n=1}^N (\ln(n+1) - \ln n) \\ &= \ln(N+1) - \ln 1 \\ &= \ln(N+1) \longrightarrow \infty. \end{aligned}$$

Thus the series of absolute values diverges, and we conclude that the series converges conditionally. $\square$

$$(k) \sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^n}$$

Solution. Set $a_n = \left| \frac{(-1)^n 3^n}{n^n} \right| = \left(\frac{3}{n} \right)^n > 0$. Then

$$\sqrt[n]{a_n} = \frac{3}{n},$$

and consequently

$$L = \lim_{n \rightarrow \infty} \sqrt[n]{a_n} = \lim_{n \rightarrow \infty} \frac{3}{n} = 0.$$

Since $L = 0 < 1$, the Root Test implies that the series converges. Therefore, the series converges absolutely $\square$

$$(l) \sum_{n=1}^{\infty} \frac{(-1)^n n}{2n-1}$$

Solution. We apply the Divergence Test. A necessary condition for convergence is that the summands approach zero. However,

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^n n}{2n-1} \right| = \lim_{n \rightarrow \infty} \frac{n}{2n-1} = \frac{1}{2} \neq 0.$$

Thus, the terms of the given series do not approach zero. By the Divergence Test, the series diverges. $\square$

3. Determine the radii and the intervals of convergence for the following power series.

(a) $\sum_{n=1}^{\infty} \frac{n^5}{5^n n!} (x-2)^n$

(b) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)} (x-1)^n$

(c) $\sum_{n=1}^{\infty} \frac{x^n}{3^n n}$

(d) $\sum_{n=1}^{\infty} 2^n n^2 x^n$

(e) $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln(n)} x^n$

(f) $\sum_{n=1}^{\infty} n! (2x-1)^n$

(g) $\sum_{n=1}^{\infty} \frac{b^n}{\ln(n)} (x-a)^n \quad b > 0$

4. Determine whether each of the statements below is true or false. If it is true, justify your response. If it is false, provide a counterexample.

(a) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty} (a_n + b_n)$ also diverges.

Solution. False; set $a_n = \frac{1}{n}$ and $b_n = -\frac{1}{n}$. □

(b) If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} (a_n)^2$ also converges.

Solution. False; set $a_n = \frac{(-1)^n}{\sqrt{n}}$. □

(c) If $a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} na_n$ diverges.

Solution. True; $a_n < na_n$ and use DCT. □

(d) If $a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \frac{a_n}{n}$ converges.

Solution. True; $\frac{a_n}{n} < a_n$ and use DCT. □

- (e) If $a_n > 0$ and the series $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty} \sin(a_n)$ converges.

Solution. True; use the series expansion for $\sin(x)$ to show that $\sin(a_n) < a_n$ and use DCT. $\square$

- (f) If $\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

Solution. False; this is not the statement of the direct comparison test. $\square$

- (g) If $\sum_{n=1}^{\infty} a_n$ conditionally converges, then $\sum_{n=1}^{\infty} |a_n|$ must diverge.

Solution. True; this is the definition of what it means for a series to conditionally converge. $\square$

- (h) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty} a_n b_n$ also diverges.

Solution. False; set $a_n = b_n = \frac{1}{n}$. $\square$

- (i) If $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{n \rightarrow \infty} a_n = 0$.

Solution. False; set $a_n = \frac{n^2}{4n^2 + 2}$. $\square$

- (j) If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} (a_n + b_n)$ converges.

Solution. False; set $a_n = \frac{1}{n!}$ and $b_n = \frac{1}{n}$. $\square$

- (k) The 37th partial sum for the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 + 5}$ is an overestimate.

Solution. False $\square$

- (l) Every convergent series must pass the ratio test.

Solution. False; set $a_n = \frac{1}{n^p}$ for $p > 1$. $\square$

- (m) If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} a_n b_n$ also converges.

Solution. False; set $a_n = b_n = \frac{(-1)^n}{\sqrt{n}}$. □

(n) If the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} (a_n)^2$ converges.

Solution. True □

5. Find a power series representation for the following functions.

(a) $f(x) = \frac{x-1}{x+2}$

Solution. First, rewrite the function as

$$\frac{x-1}{x+2} = 1 - \frac{3}{x+2} = 1 - \frac{3}{2} \frac{1}{1 + \frac{x}{2}}.$$

Recall the geometric series formula

$$\frac{1}{1-r} = \sum_{n=0}^{\infty} r^n, \quad |r| < 1.$$

Setting $r = -\frac{x}{2}$, we obtain

$$\frac{1}{1 + \frac{x}{2}} = \sum_{n=0}^{\infty} \left(-\frac{x}{2}\right)^n.$$

Therefore,

$$f(x) = 1 - \frac{3}{2} \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{2^n}.$$

The series converges when

$$\left| -\frac{x}{2} \right| < 1 \iff |x| < 2.$$

At $x = \pm 2$, the terms of the geometric series do not approach zero. Hence, the interval of convergence is

$$(-2, 2).$$

□

(b) $g(x) = \frac{x^2 + 1}{(1-x)^2}$

Solution. Recall that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad |x| < 1.$$

Differentiating both sides gives

$$\frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n.$$

Thus,

$$g(x) = (1+x^2) \sum_{n=0}^{\infty} (n+1)x^n.$$

Distributing, we obtain

$$g(x) = \sum_{n=0}^{\infty} (n+1)x^n + \sum_{n=0}^{\infty} (n+1)x^{n+2}.$$

Equivalently, combining like powers,

$$g(x) = 1 + 2x + \sum_{n=2}^{\infty} 2nx^n.$$

The series converges for $|x| < 1$. At $x = 1$, the terms do not approach zero, and at $x = -1$, the terms also fail to approach zero. Therefore, the interval of convergence is

$$(-1, 1).$$

□

(c) $h(x) = \frac{\ln(3-5x)}{x^2}$

Solution. First, rewrite the logarithm as

$$\ln(3-5x) = \ln 3 + \ln\left(1 - \frac{5x}{3}\right).$$

Recall the logarithmic power series

$$\ln(1-t) = -\sum_{n=1}^{\infty} \frac{t^n}{n}, \quad |t| < 1.$$

Setting $t = \frac{5x}{3}$ gives

$$\ln(3-5x) = \ln 3 - \sum_{n=1}^{\infty} \frac{5^n x^n}{n 3^n}.$$

Dividing by x^2 , we obtain

$$h(x) = \frac{\ln 3}{x^2} - \sum_{n=1}^{\infty} \frac{5^n x^{n-2}}{n 3^n}.$$

The logarithmic series converges when

$$\left| \frac{5x}{3} \right| < 1 \iff |x| < \frac{3}{5}.$$

At $x = -\frac{3}{5}$, the logarithmic series becomes alternating and converges. At $x = \frac{3}{5}$, it becomes a divergent harmonic series. Since the original function is undefined at $x = 0$, the representation is valid for

$$\left[-\frac{3}{5}, 0 \right) \cup \left(0, \frac{3}{5} \right).$$

□

(d) $q(x) = x \cos\left(\frac{x}{2}\right)$

Solution. Recall the Maclaurin series for $\cos(t)$:

$$\cos t = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}.$$

Setting $t = \frac{x}{2}$ gives

$$\cos\left(\frac{x}{2}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n} (2n)!}.$$

Multiplying both sides by x , we obtain

$$q(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2^{2n} (2n)!}.$$

Since the Maclaurin series for cosine converges for every real number, this series also converges for all $x \in \mathbb{R}$. Therefore, the interval of convergence is

$$(-\infty, \infty).$$

□

6. Calculate the value of the following series

(a) $\sum_{n=1}^{\infty} \frac{6^{n+2}}{3^{2n}}$

Solution. Notice that

$$\frac{6^{n+2}}{3^{2n}} = 36 \left(\frac{2}{3}\right)^n.$$

Thus,

$$\sum_{n=1}^{\infty} \frac{6^{n+2}}{3^{2n}} = 36 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n.$$

This is a geometric series with common ratio $r = \frac{2}{3}$. Using the infinite geometric series formula, we have

$$36 \sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n = 36 \left(\frac{\frac{2}{3}}{1 - \frac{2}{3}}\right) = 72.$$

□

(b) $\sum_{n=0}^{\infty} \frac{5^{n+1} + (-3)^n}{6^n}$

Solution. Splitting the sum (why can we do this?), we obtain

$$\sum_{n=0}^{\infty} \frac{5^{n+1} + (-3)^n}{6^n} = 5 \sum_{n=0}^{\infty} \left(\frac{5}{6}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{2}\right)^n.$$

Both are geometric series with common ratios of absolute value less than 1. Therefore,

$$= 5 \left(\frac{1}{1 - \frac{5}{6}}\right) + \frac{1}{1 + \frac{1}{2}} = 30 + \frac{2}{3} = \frac{92}{3}.$$

□

(c) $\sum_{n=0}^{\infty} \frac{n+1}{2^n}$

Solution. Recall the geometric series formula

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1.$$

Differentiating both sides gives

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}.$$

Reindexing the sum, we obtain

$$\sum_{n=0}^{\infty} (n+1)x^n = \frac{1}{(1-x)^2}.$$

Setting $x = \frac{1}{2}$ yields

$$\sum_{n=0}^{\infty} \frac{n+1}{2^n} = \frac{1}{\left(1 - \frac{1}{2}\right)^2} = 4.$$

□

(d) $\sum_{n=0}^{\infty} \frac{3^n}{5^n n!}$

Solution. Recall the Taylor series for the exponential function:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We can rewrite the given series as

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!}.$$

Setting $x = \frac{3}{5}$ in the exponential series gives

$$\sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}.$$

□