Series and Convergence Tests

1 Series

A **series** is simply the sum of a sequence of numbers such as

$$1 + 2 + 3 + 4 + 5$$

or

$$\log(2) + \log(4) + \log(6).$$

The examples above constitute **finite series** because there are a finite number of terms. In this class, we are interested in **infinite series**: a sum of an infinite amount of numbers. You should think of the connection between a sequence and a series as such: a series adds all the numbers in a sequence. In mathematics, we use the Greek letter Σ to denote a series. For a given sequence $\{a_n\}$, we denote the corresponding series as

$$\sum_{n=1}^{\infty} a_n := a_1 + a_2 + a_3 + a_4 + ...,$$

where $:=$ denotes a definition. We call the the number n that appears below Σ to be the **index** of the sum.

2 Convergence Tests

In this class, there are seven convergence tests you must know: the divergence test, the integral test, the direct comparison test, the limit comparison test, the alternating series test, the ratio test, and the root test. Knowing when to use each of the tests and the precise statements can be difficult to remember. Let's first go through each of the tests and state when we should use them.

1. **The Divergence Test**: This test **ONLY** tells you if the series diverges. It does **NOT** tell you if the series converges. The statement of the divergence test is:

   $$\boxed{\textit{For a series } \sum_{n=1}^{\infty} a_n\textit{, if } \lim_{n\to\infty} a_n \neq 0\textit{, then the series } \textbf{\textit{diverges}}.}$$

   The converse of the divergence statement is not true. To convince yourself, consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$. It is clear that the sequence $\dfrac{1}{n}$ goes to 0 as $n \to \infty$. However, we know that this sum does not converge from the p-series test (or the integral test). If you are ever unsure what to do and you suspect that a series diverges, this should be the first test you try.

2. **The Integral Test**: This test is used when you a_n looks like something you can integrate (examples: $\frac{1}{n\ln(n)}$ or $\frac{1}{n^2+1}$). The statement of the test is:

   > *Suppose that for a series* $\sum_{n=1}^{\infty} a_n$ *you write* $a_n = f(n)$ *and* $f(x)$ *satifies the following properties on the interval* $[1, \infty]$*:*
   >
   > (a) $f(x)$ *is positive* ($f(x) > 0$)
   >
   > (b) $f(x)$ *is continuous*
   >
   > (c) $f(x)$ *is decreasing* ($f'(x) < 0$).
   >
   > *Then, if* $\int_1^{\infty} f(x)dx$ *converges,* $\sum_{n=1}^{\infty} a_n$ *also converges and if* $\int_1^{\infty} f(x)dx$ *diverges,* $\sum_{n=1}^{\infty} a_n$ *also diverges.*

   Usually, it will be an integral that is pretty simple. I have never seen a problem that involves a very tedious integral. Assuming you are comfortable with the integration techniques we covered in this class, if you find yourself scratching your head trying to use the integral test, it is probably not the right approach. The integral test also has some hypotheses that you must satisfy before using it! You should check these on an exam if they are not obvious.

3. **The Direct Comparison Test**: This test is most commonly used to show that a series converges. It is usually used when the series is the ratio of polynomials (sometimes with square roots or cube roots, too). The statement of the test is:

   > *For a series* $\sum_{n=1}^{\infty} a_n$*, if there exists a sequence* b_n *such that* $a_n \leq b_n$ ***and*** $\sum_{n=1}^{\infty} b_n$ *converges, then* $\sum_{n=1}^{\infty} a_n$ *converges, too.*

   The test can also be used to show that a series diverges; however, it is often more difficult to bound a series from below than from above. For the sake of completeness, the test is also given as:

   > *For a series* $\sum_{n=1}^{\infty} a_n$*, if there exists a sequence* b_n *such that* $b_n \leq a_n$ ***and*** $\sum_{n=1}^{\infty} b_n$ *diverges, then* $\sum_{n=1}^{\infty} a_n$ *diverges, too.*

4. **The Limit Comparison Test**: This test is very similar to the direct comparison test. However, it is easier to show that a series diverges with this test. Again, it is used when the series is a ratio of polynomials or square and cube roots of polynomials. The statement of the test is:

> *For a series* $\sum_{n=1}^{\infty} a_n$ *such that* $a_n > 0$*, if there exists a sequence* b_n *such that* $b_n > 0$ ***and*** $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$ *and* L *is* ***finite****, then* $\sum_{n=1}^{\infty} a_n$ *behaves like* $\sum_{n=1}^{\infty} b_n$. That is, if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ also also converges; and if $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

This test is very useful for ratios of polynomials that contain square roots or cube roots. There are a few things to be mindful of when using this test. First, you want to compare the series to something that you already know converges or diverges. If you pick a b_n whose convergence you do not already know, there is no point in using this test. Usually, you will pick your b_n to be something of the form

$$\frac{1}{n^\alpha}.$$

Second, you want to be sure that the terms of both the series are always **positive**. This test would not work for an alternating series!

5. **The Alternating Series Test**: This test is used for series that look like

$$\sum_{n=1}^{\infty} (-1)^n a_n.$$

The statement of the test is:

> *For a series* $\sum_{n=1}^{\infty} (-1)^n a_n$*, if the following conditions hold:*
>
> *(a)* a_n *is decreasing (*$a_n > a_{n+1}$ *for* n *large enough)*
>
> *(b)* $\lim_{n\to\infty} a_n = 0$*,*
>
> *then the series* $\sum_{n=1}^{\infty} (-1)^n a_n$ *converges.*

This test **ONLY** proves convergence. That is, if one of the conditions in the test is not met, it does not mean that the series diverges. So, you should **NEVER** write something like "by the alternating series test, the series diverges"!! Also, this test can only be used for series that have $(-1)^n$ or $(-1)^{n+1}$; simply having terms that become negative (like $\sin(n)$) does **not** allow you to use the alternating series test.

6. **The Ratio Test**: The ratio test allows you to determine if a series that has **powers**, **exponentials**, or **factorials**. The statement of the test is:

   For a series $\sum_{n=1}^{\infty} a_n$, *calculate the following quantity:*

   $$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$$

   If

   (a) $L < 1$, *the series* ***converges absolutely***

   (b) $L > 1$, *the series* ***diverges***

   (c) $L = 1$, *the test is inconclusive. You cannot say anything about the series and must try a different test.*

   This test can be used on almost anything: polynomials, alternating series, exponentials, factorials, etc. You should usually resort to this test if your series includes many different terms like factorials and exponentials or exponentials and powers. Note that we can use the ratio test on an alternating series if the alternating series test seems too difficult to employ. The absolute values in the limit will negate any oscilations between 1 and -1 in the alternating series.

   If you are ever completely stuck on what to do, you can try this test because it will always spit out an answer. As long as the limit is not equal to one, you will be able to provide an answer on an exam if you use this test.

7. **The Root Test**: The root test is usually only used in very specific circumstances: when the entire series is raised to the power n. The statement of the test is:

> *For a series* $\sum_{n=1}^{\infty} a_n$, *calculate the following quantity:*
>
> $$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$
>
> *If*
>
> *(a)* $L < 1$, *the series* ***converges absolutely***
>
> *(b)* $L > 1$, *the series* ***diverges***
>
> *(c)* $L = 1$, *the test is inconclusive. You cannot say anything about the series and must try a different test.*

Note that the conditions are the same as those of the ratio test. This test you will probably use the least. It is really only useful for series that have forms like

$$\left(\frac{n^3}{n^7+1}\right)^n \quad \text{or} \quad \left(\frac{\cos(n)}{n^2+7}\right)^n$$

If you want a concise version of above, here is a table that you can reference.

If the series looks like...	**Use this test...**
The limit of a_n does **not** go to zero	Divergence Test
Terms that are like $\dfrac{\text{polynomial of } n}{\text{polynomial of } n}$ or $\dfrac{\sqrt[p]{\text{polynomial of } n}}{\sqrt[q]{\text{polynomial of } n}}$	Limit Comparison Test
Terms involve $n!$, n^k, a^n, or n^n	Ratio Test
Terms are raised to the n-th power, e.g., $(a_n)^n$	Root Test
Terms look like a function $f(n)$ that **can** be integrated	Integral Test
Terms are of the form $1/n^p$	p-Series Test
Terms that have $(-1)^n$	First use alternating series. If the limit is hard to calculate, try the ratio test
Terms that can be bounded **above** by $\dfrac{1}{n^p}$	Direct Comparison Test