Series Practice Solutions

Determine if the following series converge or diverge.

For the solutions, I will make some remarks about how to reason through each problem. Then, I will write the solution as a "proof". These are mathematical arguments you are making; you should have some structure to them.

1. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+6}$

   Claim: The series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+6}$ converges.

   Proof. Let $a_n = \dfrac{1}{n^3+6}$. Consider the following string of inequalities:

   $$n^3 + 6 > n^3 \Rightarrow \frac{1}{n^3} < \frac{1}{n^3+6} = a_n.$$

   Hence, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3} < \sum_{n=1}^{\infty} a_n$. The former series converges via the p-test using $p = 3$. The Direct Comparison Test then yields that $\displaystyle\sum_{n=1}^{\infty} a_n$ also converges. □

2. $\displaystyle\sum_{n=8}^{\infty} \frac{\ln^2(n)}{n}$

   Remarks: Whenever the series has a $1/n$ and $\ln(n)$ terms, its usually going to require the integral test.

   Claim: The series $\displaystyle\sum_{n=8}^{\infty} \frac{\ln^2(n)}{n}$ diverges.

   Proof. Let $a_n = \dfrac{\ln^2(n)}{n}$ and set $f(n) = a_n$. To use the integral test, we must show that $f(x)$ is positive, continuous, and decreasing on the interval $I = [8, \infty)$.
   f is positive: On the interval I, the natural log, and hence its square, and $1/x$ are positive. Hence, their product is also positive. Thus, f is positive on I.
   f is continuous: On the interval I, the natural log, and hence its square, and $1/x$ are continuous. Hence, their product is also continuous. Thus, f is continuous on I.
   f is decreasing: Consider the derivative of f given by

   $$f'(x) = \frac{2\ln(x)\frac{1}{x} \cdot x - 1 \cdot \ln^2(x)}{x^2} = \frac{\ln(x)(2 - \ln(x))}{x^2}.$$

On the interval I, $(2-\ln(x)) < 0$ and $\ln(x) > 0$. It follows that $f'(x) < 0$ on I. Thus, f is decreasing on I.

We have now satisfied all the hypotheses to apply the integral test. We calculate

$$\int_8^\infty f(x)dx = \int_8^\infty \frac{\ln^2(x)}{x}dx \underset{u=\ln(x)}{=} \int_{\ln(8)}^\infty u^2 du = \lim_{c\to\infty}\left[\frac{u^3}{3}\right]_{\ln(8)}^c \to \infty.$$

As the integral of $f(x)$ on I does not converge, the integral test provides that $\sum_{n=8}^\infty a_n$ also diverges. □

3. $\sum_{n=1}^\infty \frac{n}{\sqrt{n^3+n^2+1}}$

Remarks: When you have a ratio of polynomials, it's a good strategy to compare the highest orders of the numerators and denominators. The order of the numerator is clearly 1. The order of the denominator is 3/2. Thus, the series is likely to behave like the series $1/n^{1/2}$. Since the series of $1/n^{1/2}$ diverges, we should use the limit comparison test with $b_n = 1/n^{1/2}$.

Claim: The series $\sum_{n=1}^\infty \frac{n}{\sqrt{n^3+n^2+1}}$ diverges.

Proof. Let $a_n = \frac{n}{\sqrt{n^3+n^2+1}}$. Consider the sequence $b_n = \frac{1}{n^{1/2}}$. Note that both $a_n, b_n > 0$, so we can use the Limit Comparison Test. We calculate

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^{3/2}}{\sqrt{n^3+n^2+1}} = \lim_{n\to\infty}\sqrt{\frac{n^3}{n^3+n^2+1}} = \sqrt{\lim_{n\to\infty}\frac{n^3}{n^3+n^2+1}} = \sqrt{1} = 1,$$

where the third equality holds because the square root function is continuous for all positive real numbers. As the series $\sum_{n=1}^\infty b_n$ by the p-series test using $p = 1/2$, the Limit Comparison Test yields that $\sum_{n=1}^\infty a_n$ also diverges. □

4. $\sum_{n=1}^\infty (-1)^n \frac{n^3}{7^{n+7}}$

This series is alternating since it contains the terms $(-1)^n$. It follows that we can use the alternating series test of the ratio test. Because the series contains a 7^{n+7} term, we should use the ratio test.

Claim: The series $\sum_{n=1}^\infty (-1)^n \frac{n^3}{7^{n+7}}$ converges.

Proof. Let $a_n = (-1)^n \frac{n^3}{7^{n+7}}$. We calculate the value $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ to be

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(-1)^{n+1}(n+1)^3}{7^{n+8}} \cdot \frac{7^{n+7}}{(-1)^n n^3}\right| = \lim_{n\to\infty} \frac{n^3}{7(n+1)^3} = \frac{1}{7}.$$

As $L < 1$, the Ratio Test provides that $\sum_{n=1}^{\infty} a_n$ converges. □

5. $\sum_{n=1}^{\infty} \frac{n^3 + 4n^2 - 3n + 8}{8n^2 - 3n^3 + 6n - 47}$

Remarks: This problem is similar to problem 3. We see the orders of the numerator and denominator are both 3. We know from the second exam that the limit is the ratio of the coefficients of n^3, which is not zero.

Claim: The series $\sum_{n=1}^{\infty} \frac{n^3 + 4n^2 - 3n + 8}{8n^2 - 3n^3 + 6n - 47}$ diverges

Proof. Let $a_n = \frac{n^3 + 4n^2 - 3n + 8}{8n^2 - 3n^3 + 6n - 47}$. Consider the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^3 + 4n^2 - 3n + 8}{8n^2 - 3n^3 + 6n - 47} = -\frac{1}{3}.$$

As the limit of a_n is nonzero, the Divergence Test provides that $\sum_{n=1}^{\infty} a_n$ diverges. □

6. $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^n}$

This problem is a standard ratio test example because the series includes exponential and factorial terms.

Claim: The series $\sum_{n=1}^{\infty} \frac{(n+1)!}{n^n}$ converges.

Proof. Let $a_n = \frac{(n+1)!}{n^n}$. We calculate the value $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$ to be

$$\begin{aligned} L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \left|\frac{(n+2)!}{(n+1)^{n+1}} \cdot \frac{n^n}{(n+1)!)}\right| = \lim_{n\to\infty} \left|\frac{(n+2)(n+1)!}{(n+1)^n(n+1)} \cdot \frac{n^n}{(n+1)!)}\right| \\ &= \lim_{n\to\infty} \left|\frac{(n+2)(n+1)!}{(n+1)^n(n+1)} \cdot \frac{n^n}{(n+1)!)}\right| = \lim_{n\to\infty} \left|\frac{n^n(n+2)}{(n+1)^n(n+1)}\right| = \lim_{n\to\infty} \left|\left(\frac{n}{n+1}\right)^n \frac{(n+2)}{(n+1)}\right| \\ &= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \frac{(n+2)}{(n+1)} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \lim_{n\to\infty} \frac{(n+2)}{(n+1)} = \frac{1}{e} \cdot 1 = \frac{1}{e}. \end{aligned}$$

As $L < 1$, the Ratio Test provides that $\sum_{n=1}^{\infty} a_n$ converges. □

7. $\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)}$

Remarks: There are no terms in the numerator, that is a good sign that the series is likely to converge. There is also an exponential in the denominator. As exponentials grow faster than any polynomial, you should have the idea that this series probably converges. Further, exponentials indicate this series is a good candidate to use the ratio test.

Claim: The series $\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)}$ converges.

Proof. Let $a_n = \frac{1}{n(e^{2\pi n} - 1)}$. First, we have that

$$n\left(e^{2\pi n} - 1\right) > \left(e^{2\pi n} - 1\right) \Rightarrow \frac{1}{n\left(e^{2\pi n} - 1\right)} < \frac{1}{\left(e^{2\pi n} - 1\right)}.$$

Let $b_n = \frac{1}{(e^{2\pi n} - 1)}$. We want to show that $\sum_{n=1}^{\infty} b_n$ converges. We calculate the value $\lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right|$ to be

$$L = \lim_{n\to\infty} \left|\frac{b_{n+1}}{b_n}\right| = \lim_{n\to\infty} \left|\frac{1}{e^{2\pi n + 2\pi} - 1} \cdot \frac{e^{2\pi n} - 1}{1}\right| = \lim_{n\to\infty} \frac{e^{2\pi n} - 1}{e^{2\pi}e^{2\pi n} - 1} = \frac{1}{e^{2\pi}}$$

As $L < 1$, the Ratio Test provides $\sum_{n=1}^{\infty} b_n$ converges. It now follows from the Direct Comparison Test that $\sum_{n=1}^{\infty} a_n$ also converges. □

This series has a nice value:

$$\sum_{n=1}^{\infty} \frac{1}{n(e^{2\pi n} - 1)} = \ln\left(\sqrt[4]{2}\right) - \frac{1}{8}\ln(\pi) + \frac{1}{2}\ln\left(\Gamma\left(\frac{3}{4}\right)\right) - \frac{\pi}{24}$$

where

$$\Gamma\left(\frac{3}{4}\right) = \int_0^{\infty} x^{-1/4} e^{-x} dx.$$

8. $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$

Remarks: At first glance, it does not seem like any of the tests we have could handle this. If you ever hit a point like this, you should always just take the limit to see if it does not go to zero. This will be the case for this problem.

Claim: The series $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+1}\right)$ diverges.

Proof. Let $a_n = \ln\left(\frac{n}{2n+1}\right)$. Consider the limit

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{n}{2n+1}\right) = \ln\left(\lim_{n\to\infty} \frac{n}{2n+1}\right) = \ln\left(\frac{1}{2}\right),$$

where the second equality because the natural log function is continuous for all positive real numbers. As the limit of a_n is nonzero, the Divergence Test provides that $\displaystyle\sum_{n=1}^{\infty} a_n$ diverges. □

9. $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{5^n n!}$

Remarks: The terms of the series include exponentials and factorials. The only test we have to deal with these is the ratio test. When you see a series with terms like this, you should use the ratio test.

Claim: The series $\displaystyle\sum_{n=1}^{\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{5^n n!}$ converges.

Proof. Let $a_n = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{5^n n!}$. We calculate the value $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}|$ to be

$$\begin{aligned} L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty} \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{5^{n+1}(n+1)n!} \cdot \frac{5^n n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} \\ &= \lim_{n\to\infty} \frac{(2n+1)}{5(n+1)} = \frac{2}{5} \end{aligned}$$

As $L < 1$, the Ratio Test provides that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges. □