Math 1301 Practice Final Exam Version 2

Vanderbilt University

8 December 2025

Name: Key

Please do not open the exam until instructed to do so.
You are allowed a one-page (double-sided) formula sheet.
Your instructor may ask to see your formula sheet.
No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

Part 1. *(20 points)*

Q1 $\frac{d}{dx}\left[\log_9(5^{\sin(e^x)}) + \cos^{-1}(\ln(\sqrt[50]{x}))\right] =$ $\boxed{\log_9(5)\cos(e^x)e^x - \frac{1}{x\sqrt{1-1/50^2\ln^2(x)}}\cdot\frac{1}{50}}$

$$\log_9(5^{\sin(e^x)}) + \cos^{-1}(\ln(\sqrt[50]{x}))$$
$$= \sin(e^x)\log_9(5) + \cos^{-1}\left(\frac{1}{50}\ln(x)\right)$$

Q2 $\frac{d}{dx}\ln^{5^x}(x) =$ $\boxed{\ln^{5^x}(x)\left[5^x\ln(5)\ln(\ln(x)) + \frac{5^x}{x\ln(x)}\right]}$

$$y = \ln^{5^x}(x)$$
$$\ln(y) = 5^x\ln(\ln(x))$$
$$\frac{1}{y}\frac{dy}{dx} = 5^x\ln(5)\ln(\ln(x)) + \frac{5^x}{x\ln(x)}$$
$$\frac{dy}{dx} = \ln^{5^x}(x)\left[5^x\ln(5)\ln(\ln(x)) + \frac{5^x}{x\ln(x)}\right]$$

Q3 $\lim_{x\to 0^+}\left(\frac{\sin(x)}{x}\right)^{1/x^2} =$ $\boxed{1/e^{1/6}}$

$$\left(\frac{\sin(x)}{x}\right)^{1/x^2} = e^{\frac{1}{x^2}\ln(\sin(x)) - \frac{1}{x^2}\ln(x)}$$

$$\lim_{x\to 0^+}\frac{\ln(\sin(x)) - \ln(x)}{x^2} \overset{L'H}{=} \lim_{x\to 0^+}\frac{\cot(x) - 1/x}{2x} = \lim_{x\to 0^+}\frac{x\cot(x) - 1}{2x^2}$$

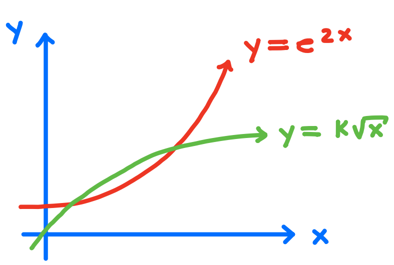
$$\overset{L'H}{=} \lim_{x\to 0^+}\frac{\cot(x) - x\csc^2(x)}{4x} \overset{L'H}{=} \lim_{x\to 0^+}\frac{1}{4}\left(-\csc^2(x) - \csc^2(x) + 2x\cot(x)\csc^2(x)\right)$$

$$= \lim_{x\to 0^+}\frac{1}{4}\left(2x\frac{\cos(x)}{\sin(x)} - 2\right)\frac{1}{\sin^2(x)} = \lim_{x\to 0^+}\frac{2x\cos(x) - 2\sin(x)}{4\sin^3(x)} \overset{L'H}{=} \lim_{x\to 0^+}\frac{-2x\sin(x)}{12\sin^2(x)\cos(x)}$$

$$= -\frac{1}{6}\lim_{x\to 0^+}\frac{x}{\sin(x)\cos(x)} \overset{L'H}{=} -\frac{1}{6}\lim_{x\to 0^+}\frac{1}{\cos^2(x) - \sin^2(x)} = -1/6.$$ As e^x is continuous,

$$\lim_{x\to 0^+}\left(\frac{\sin(x)}{x}\right)^{1/x^2} = \frac{1}{e^{1/6}}$$

Q4 The value of k such that the equation $e^{2x} = k\sqrt{x}$ has *one* solution is $\boxed{k = 2\sqrt{e}}$



Key idea: One solution means derivatives are also equal!

$$e^{2x} = k\sqrt{x}$$
$$2e^{2x} = \frac{k}{2\sqrt{x}}$$
$$\Rightarrow \quad e^{2x} = 4xe^{2x} \Rightarrow x = 1/4$$
$$e^{1/2} = k\frac{1}{2} \Rightarrow k = 2\sqrt{e}$$

Part 2. (*10 points*) Consider the two power series given by

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2} \qquad \text{and} \qquad J_1(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n+1}}{n!(n+1)!2^{2n+1}}$$

These are called Bessel Functions of the first kind.

Q1 Determine the radii of convergence of $J_0(x)$ and $J_1(x)$.

i) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 4^n}$

Then, $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n!)^2 4^n}{(n+1)^2 (n!) 4^{n+1}} x^2 \right| = \lim_{n\to\infty} \frac{1}{4(n+1)^2} x^2 = 0 \quad \forall x \in \mathbb{R}.$

ROC $= \infty$.

ii) Same idea for $J_1(x)$ to see ROC $= \infty$

Q2 Show that $y(x) = J_0(x)$ solves the differential equation

$$x^2 y''(x) + xy'(x) + x^2 y(x) = 0.$$

$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(n!)^2 4^n} = y(x)$ Set $a_n = \frac{(-1)^n}{(n)! 4^n}$

i) $x^2 y(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{(n!)^2 4^n}$

ii) $x y'(x) = x \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n-1}}{(n!)^2 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n) x^{2n}}{(n!)^2 4^n}$

iii) $x^2 y''(x) = x^2 \sum_{n=0}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(n!)^2 4^n} = \sum_{n=0}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n}}{(n!)^2 4^n}$

Summing (ii) and (iii) $= \sum_{n=0}^{\infty} 4n^2 a_n x^{2n}$

Summing the above result with (i) yields

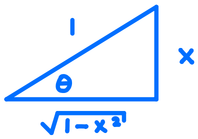
$\underbrace{\sum_{n=0}^{\infty} a_n x^{2n+2}}_{= \sum_{n=1}^{\infty} a_{n-1} x^{2n}} + \sum_{n=0}^{\infty} 4n^2 a_n x^{2n}$

hence consider $4n^2 a_n + a_{n-1}$

$= \frac{4n^2 (-1)^n}{(n!)^2 4^n} + \frac{(-1)^{n-1}}{((n-1)!)^2 4^{n-1}} = 0.$

Part 3 (*15 points*) Evaluate the following integrals.

Q1 $\int_0^1 \tan(\sin^{-1}(x)) =$ $\boxed{1}$



$\tan(\sin^{-1}(x)) = \frac{x}{\sqrt{1-x^2}}$

$\int \tan(\sin^{-1}(x))\,dx = \int \frac{x\,dx}{\sqrt{1-x^2}}$

$u = 1-x^2$

$du = -2x\,dx$

$= -\frac{1}{2}\int u^{-1/2}\,du = -u^{1/2} = -\sqrt{1-x^2}$

$\int_0^1 \tan(\sin^{-1}(x))\,dx = -\sqrt{1-x^2}\Big|_0^1 = -(0-1) = 1$

Q2 $\int \sqrt{1-x^2}\,dx =$ $\boxed{\frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x)}$

$\int \sqrt{1-x^2}\,dx$

$u = \sqrt{1-x^2}$ $\quad dv = dx$

$du = \frac{-x}{\sqrt{1-x^2}}\,dx$ $\quad v = x$

$= x\sqrt{1-x^2} + \int \frac{x^2}{\sqrt{1-x^2}}\,dx$

$\int \frac{x^2}{\sqrt{1-x^2}}\,dx = \int \frac{x^2-1+1}{\sqrt{1-x^2}}\,dx$

$= -\int \frac{1-x^2}{\sqrt{1-x^2}}\,dx + \int \frac{dx}{\sqrt{1-x^2}}$

$= -\int \sqrt{1-x^2}\,dx + \arcsin(x)$

$\Rightarrow \underbrace{\int \sqrt{1-x^2}\,dx = -\int \sqrt{1-x^2}\,dx}_{\text{combine to } 2\int\sqrt{1-x^2}\,dx} + x\sqrt{1-x^2} + \arcsin(x) \Rightarrow \int \sqrt{1-x^2}\,dx = \frac{1}{2}x\sqrt{1-x^2} + \frac{1}{2}\arcsin(x)$

Q3 $\int \frac{\cot(x)}{\ln(\sin(x))}\,dx =$ $\boxed{\ln(\ln(\sin(x)))}$

$\int \frac{\cot(x)}{\ln(\sin(x))}\,dx = \int \frac{\cos(x)}{\sin(x)\ln(\sin(x))}\,dx$

$u = \sin(x)$

$du = \cos(x)\,dx$

$= \int \frac{du}{u\ln(u)}$

$s = \ln(u)$

$ds = \frac{du}{u}$

$= \int \frac{ds}{s} = \ln(s) = \ln(\ln(\sin(x)))$

Part 4. (*15 points*)

Q1 If $f(x) = \sum_{n=1}^{\infty} a_n x^n$, then $\sum_{n=1}^{\infty} n^2 a_n x^n =$ $\boxed{xf'(x) + x^2 f''(x)}$

$$f(x) = \sum_{n=1}^{\infty} a_n x^n, \quad f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} \Rightarrow xf'(x) = \sum_{n=1}^{\infty} n a_n x^n$$

$$\Rightarrow \frac{d}{dx}(xf'(x)) = f'(x) + xf''(x) = \sum_{n=1}^{\infty} n^2 a_n x^{n-1}$$

$$\Rightarrow xf'(x) + x^2 f''(x) = \sum_{n=1}^{\infty} n^2 a_n x^n$$

Q2 The values of p and q such that $\sum_{n=3}^{\infty} \frac{1}{n^p \ln^q(n)}$ converges are $\boxed{p=1,\ q>1 \quad p>1,\ q\in\mathbb{R}}$

$\underline{p=1}$ $\sum_{n=3}^{\infty} \frac{1}{n \ln^q(n)}$

$$\int_3^{\infty} \frac{dx}{x \ln^q(x)} = \int_{\ln(3)}^{\infty} \frac{dx}{u^q}$$

converges if and and only if $q>1$.

$\boxed{p=1,\ q>1}$

$\underline{p>1}$ $n^p \ln^q(n) > n^p$

$$\Rightarrow \frac{1}{n^p \ln^q(n)} < \frac{1}{n^p}$$

$\downarrow$

converges via p-series

Direct comparison test yields convergence for

$\boxed{p>1,\ q\in\mathbb{R}}$

$\underline{p<1}$ $\sum_{n=3}^{\infty} \frac{1}{n^p \ln^q(n)}$

$$\int_3^{\infty} \frac{dx}{x^p \ln^q(x)}$$

$u = \ln(x) \Rightarrow x = e^u$

$du = \frac{dx}{x}$

$$= \int_{\ln(3)}^{\infty} \frac{e^{(1-p)u}}{u^q} du \longrightarrow +\infty \quad \forall p\in(0,1),\ q\in\mathbb{R}$$

Q3 Evaluate the following integral: $\int \frac{\ln(\ln(x))}{x} dx =$ $\boxed{\ln(x)\ln(\ln(x)) - \ln(x)}$

$$\int \frac{\ln(\ln(x))}{x} dx$$

$t = \ln(x)$

$dt = \frac{dx}{x}$

$$= \int \ln(t) dt$$

$u = \ln(t) \quad dv = dt$

$du = \frac{dt}{t} \quad v = t$

$$= t\ln(t) - \int dt$$

$$= t\ln(t) - t$$

$$= \ln(x)\ln(\ln(x)) - \ln(x)$$

Part 5. (*15 points*)

Q1 Consider the parametric equations

$$x(t) = e^t \cos(t) \quad y(t) = e^t \sin(t)$$

Eliminate the parameter and find $\frac{dy}{dx}$ in terms of t.

i) $x^2 + y^2 = e^{2t}\cos^2(t) + e^{2t}\sin^2(t)$
$= e^{2t}$
$\Rightarrow t = \frac{1}{2}\ln(x^2+y^2)$

ii) $\frac{x}{y} = \frac{e^t\cos(t)}{e^t\sin(t)} = \cot(t)$
$\Rightarrow t = \cot^{-1}(x/y)$

$\cot^{-1}(x/y) = \frac{1}{2}\ln(x^2+y^2)$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{e^t\sin(t) + e^t\cos(t)}{e^t\cos(t) - e^t\sin(t)}$
$= \frac{y+x}{x-y}$

Q2 Find the length of the polar curve of $r = \frac{1}{\theta}$ from $\cancel{0}$ $\pi \le \theta \le 2\pi$

$r(\theta) = \frac{1}{\theta}$ $\Rightarrow r^2 + (r'(\theta))^2 = \frac{1}{\theta^2} + \frac{1}{\theta^4}$

$r'(\theta) = \frac{-1}{\theta^2}$ $\sqrt{\frac{1}{\theta^2}+\frac{1}{\theta^4}} = \frac{\sqrt{\theta^2+1}}{\theta^2}$

$\int \sqrt{r^2 + (r'(\theta))^2}\, d\theta$

$= \int \frac{\sqrt{\theta^2+1}}{\theta^2} d\theta = -\frac{\sqrt{1+\theta^2}}{\theta} + \int \frac{d\theta}{\sqrt{1+\theta^2}}$

$u = \sqrt{1+\theta^2}$ $dv = \frac{d\theta}{\theta^2}$

$du = \frac{\theta\, d\theta}{\sqrt{1+\theta^2}}$ $v = -\theta^{-1}$

$\boxed{= \ln(\theta + \sqrt{1+\theta^2}) - \frac{\sqrt{1+\theta^2}}{\theta}}$

$\int \frac{d\theta}{\sqrt{1+\theta^2}} = \int \frac{d\theta}{\theta\sqrt{1/\theta^2+1}}$

$t^2 = 1 + 1/\theta^2 \Rightarrow \theta^2 = \frac{1}{t^2-1}$

$2t\,dt = -2\frac{d\theta}{\theta^3} \Rightarrow \frac{d\theta}{\theta} = -t\theta^2 dt$

Substituting yields

$= \int \frac{dt}{1-t^2} = \int \frac{A}{1-t}dt + \int \frac{B}{1+t}dt$

$= \frac{-1}{2}\int \frac{dt}{1-t} + \frac{1}{2}\int\frac{dt}{1+t} = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right|$

$\frac{1+t}{1-t} \cdot \frac{(1+t)}{(1+t)} = \frac{(1+t)^2}{1-t^2} = -\theta^2(1+t^2)$

$= \ln(\theta(1+t)) = \ln(\theta + \theta\sqrt{1+1/\theta^2})$

$= \ln(\theta + \sqrt{1+\theta^2})$.

Q3 Eliminate the parameter of the parametric equations $x = 2\cos(\theta)$, $y = 1 + \sin(\theta)$. Find the points of the horizontal and vertical tangent lines.

$x = 2\cos(\theta) \Rightarrow \cos(\theta) = x/2$

$y = 1 + \sin(\theta) \Rightarrow \sin(\theta) = y - 1$

Hence, $x^2/4 + y^2 - 2y + 1 = 1$.

$\frac{dx}{d\theta} = -2\sin(\theta)$ $\frac{dy}{d\theta} = \cos(\theta)$

$-2\sin(\theta) = 0$ $\cos(\theta) = 0$

$\theta = 0, \pi$ $\theta = \pi/2, 3\pi/2$

Part 6. (*10 points*)

Q1 The general solution to $\dfrac{dy}{dx} = \dfrac{3^x y\left(\ln^2(y) + 3\ln(y) + 2\right)}{1+9^x}$ is $\boxed{\dfrac{\ln(y)+1}{\ln(y)+2} = e^{\tan^{-1}(3^x)} + C}$

$$\frac{dy}{dx} = \frac{3^x y(\ln^2(y)+3\ln(y)+2)}{1+9^x}$$

$$\int \frac{dy}{y(\ln^2(y)+3\ln(y)+2)} = \int \frac{3^x}{1+(3^x)^2}\,dx$$

$u_1 = \ln(y)$ $\qquad u_2 = 3^x$

$du_1 = \frac{1}{y}\,dy$ $\qquad du = \ln(3)3^x\,dx$

$$\frac{1}{(u_1+2)(u_1+1)} = \frac{A}{u_1+2} + \frac{B}{u_1+1}$$

$A = -1$, $B = 1$

$$\int \frac{du_1}{(u_1+1)(u_1+2)} = \frac{1}{\ln(3)} \int \frac{du_2}{1+(u_2)^2}$$

$$\int \frac{-1}{u_1+2} + \frac{1}{u_1+1}\,du = \frac{1}{\ln(3)} \int \frac{du_2}{1+(u_2)^2}$$

$$-\ln(u_1+2) + \ln(u_1+1) = \frac{1}{\ln(3)} \tan^{-1}(u_2) + C$$

$$-\ln(\ln(y)+2) + \ln(\ln(y)+1) = \frac{1}{\ln(3)} \tan^{-1}(3^x) + C$$

Q2 Use power series to evaluate: $\displaystyle\lim_{x\to 0} \frac{1-\cos(x)}{1+x-e^x} = \boxed{-1}$

$$\lim_{x\to 0} \frac{1-\cos(x)}{1+x-e^x} = \lim_{x\to 0} \frac{1-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n)!}}{1+x+\sum_{n=0}^{\infty}\frac{x^n}{n!}} = \lim_{x\to 0} \frac{1-\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots\right)}{1+x-\left(1+x+x^2/2+\dots\right)}$$

$$= \lim_{x\to 0} \frac{x^2/2 - x^4/4! + \dots}{-x^2/2 - x^3/3! - x^4/4! - \dots} \overset{L'H}{=} \lim_{x\to 0} \frac{x - x^3/3! + x^5/5! - \dots}{-x - x^2/2! - x^3/3! - \dots} \overset{L'H}{=} \lim_{x\to 0} \frac{1 - x^2/2! + x^4/4! - \dots}{-1 - x - x^2/2! - \dots}$$

$= -1$

Part 7 (*20 points*) Determine if the following series converge or diverge. Show your work and state any necessary hypotheses.

Q1 $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n\sqrt{n^2+1}}$

Method 1 (Direct Comparison Test)

$n^2+1 > n^2$

$\Rightarrow \sqrt{n^2+1} > n$

$\Rightarrow n\sqrt{n^2+1} > n^2$

$\Rightarrow \frac{1}{n\sqrt{n^2+1}} < \frac{1}{n^2}$

As $\frac{1}{n^2}$ converges via the p-series test, the direct comparison test yields $\frac{1}{n\sqrt{n^2+1}}$ converges, too.

Method 2 (Limit Comparison Test)

Set $a_n = \frac{1}{n\sqrt{n^2+1}}$ and $b_n = \frac{1}{n^2}$. As $\lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$, the limit comparison test yields that $\sum a_n$ converges as $\sum b_n$ converges.

Q2 $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^n}{n^{4n}}$

set $a_n = \frac{(n!)^n}{n^{4n}}$. Then, $\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)^n (n!)^n}{(n+1)^{4n+4}} \cdot \frac{n^{4n}}{(n!)^n}\right|$

$= \left|\frac{n^{4n}}{(n+1)^{3n+4}}\right| = \left|\frac{n^n \cdot n^{3n}}{(n+1)^4 (n+1)^{3n}}\right| = \underbrace{\frac{n^n}{(n+1)^4}}_{\to\infty} \underbrace{\left(\frac{n}{n+1}\right)^{3n}}_{\to 1/e^3}$

By the ratio test yields divergence

Q3 $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{2^n}$ (Use the alternating series test)

Set $a_n = \frac{n^2}{2^n}$ and consider $f(x) = \frac{x^2}{2^x}$. Then, $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{x^2}{2^x}$ $\overset{L'H}{=} \lim_{x\to\infty} \frac{2x}{\ln(2)2^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{2}{\ln^2(2)\,2^x} = 0$. Hence, $\lim_{n\to\infty} a_n = 0$. Thus, the Alternating Series Test yields convergence.

Q4 $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{6}\right)$

$\lim_{n\to\infty} \sin\left(\frac{n\pi}{6}\right) = DNE$. By the divergence test $\sum \sin\left(\frac{n\pi}{6}\right)$ diverges.

Part 8 (*20 points*) Define the function $f(x) = \displaystyle\int_0^x \frac{1}{(t+1)\sqrt{1-\ln^2(t+1)}}\, dt$

Q1 The domain of $f(x)$ is $\boxed{\{x \in \mathbb{R} \mid e^{-1}-1 \le x \le e-1\}}$

$$f(x) = \int_0^x \frac{dt}{(t+1)\sqrt{1-\ln^2(t+1)}} \overset{u=t+1}{=} \int_0^x \frac{du}{u\sqrt{1-\ln^2(u)}} \overset{s=\ln(u)}{=} \int_0^x \frac{ds}{\sqrt{1-s^2}} = \sin^{-1}(\ln(x+1))$$

$$f(x) = \arcsin(\ln(t+1))$$

$$-1 \le \ln(x+1) \le 1$$

$$e^{-1}-1 \le x \le e-1$$

Q2 $f(x)$ has an inverse on $\boxed{\{x \in \mathbb{R} \mid e^{-1}-1 \le x \le e-1\}}$

$$f'(x) = \frac{1}{(x+1)\sqrt{1-\ln^2(x+1)}} = \underbrace{\frac{1}{x+1}}_{<0} \cdot \underbrace{\frac{1}{\sqrt{1-\ln^2(x+1)}}}_{<0}$$

Q3 $f\left(\displaystyle\sum_{n=1}^{\infty} \frac{(3)^{n/2}}{n!\ 2^n}\right) = \boxed{\frac{\pi}{3}}$

$$\sum_{n=0}^{\infty} \frac{3^{n/2}}{n!\,2^n} = \sum_{n=0}^{\infty} \frac{(\sqrt{3}/2)^n}{n!} = e^{\sqrt{3}/2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{3^{n/2}}{n!\,2^n} = e^{\sqrt{3}/2} - 1$$

$$f(e^{\sqrt{3}/2} - 1) = \arcsin(\ln(e^{\sqrt{3}/2} - 1 + 1))$$

$$= \sin^{-1}(\sqrt{3}/2) = \pi/3$$

Q4 $(f^{-1})'(1/2) = \boxed{e^{\sin(1/2)}\cos(1/2)}$

$$(f^{-1})'\left(\tfrac{1}{2}\right) = \frac{1}{f'(f^{-1}(1/2))} = e^{\sin(1/2)}\sqrt{1-\sin^2(1/2)} = e^{\sin(1/2)}\cos(1/2)$$

$$\text{arsin}(\ln(x+1)) = 1/2$$

$$\ln(x+1) = \sin(1/2)$$

$$x = e^{\sin(1/2)} - 1$$

Part 9 (*20 points*) Suppose that $y = f(x)$ is a solution to the differential equation

$$\frac{dy}{dx} = yx\ln(x) + k\sin(x)$$

where $k > 0$ and $f(1) = 4$.

Q1 The second order Taylor polynomial of $f(x)$ at $x = 1$ is given by $\boxed{f(x) \approx 4 + k\sin(1)(x-1) + \frac{1}{2}(4 + k\cos(1))(x-1)^2}$

$f(1) = 4$

$f'(x) = \frac{dy}{dx} = yx\ln(x) + k\sin(x)$

$f'(1) = (4)(1)\ln(1) + k\sin(1) = k\sin(1)$

$f''(x) = \frac{dy}{dx}x\ln(x) + y(\ln(x) + x) + k\cos(x)$

$f''(1) = (k\sin(1))(1)\ln(1) + 4(\ln(1) + 1) + k\cos(1) = 4 + k\cos(1)$

$f(x) \approx 4 + k\sin(1)(x-1) + \frac{1}{2}(4 + k\cos(1))(x-1)^2$

Q2 If $k = 0$, then the solution to the initial value problem is given by $\boxed{y = \exp\left(\frac{x^2}{2}(\ln(x) - \frac{1}{2}) + \ln(4) + 1/4\right)}$

If $k = 0$, $\frac{dy}{dx} = yx\ln(x)$

$\frac{dy}{y} = x\ln(x)\,dx$

$\int \frac{dy}{y} = \int x\ln(x)\,dx$

$u = \ln(x)$ $\quad dv = x\,dx$

$du = \frac{dx}{x}$ $\quad v = \frac{x^2}{2}$

$\ln|y| = \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\,dx$

$\ln|y| = \frac{x^2}{2}(\ln(x) - \frac{1}{2}) + c$

$f(1) = 4$

$\ln(4) = \frac{1}{4}(-1) + c$

$c = \ln(4) + 1/4$

$\ln|y| = \frac{x^2}{2}(\ln(x) - \frac{1}{2}) + \ln(4) + 1/4$

$y = \exp\left(\frac{x^2}{2}(\ln(x) - \frac{1}{2}) + \ln(4) + 1/4\right)$

Q3 The domain of $f(x)$ is $\boxed{x > 0}$

Part 10 (*25 points*) Define the function $g(x) = xe^x$.

Q1 Calculate $\int_0^1 g(x)\,dx$ in two ways: an integration techique and a power series technique.

i) $\int_0^1 xe^x\,dx$

$u = x \quad dv = e^x dx$

$du = dx \quad v = e^x$

$= xe^x\big|_0^1 - \int_0^1 e^x dx$

$= e - (e-1) = 1$

ii) $xe^x = x\sum_{n=0}^{\infty}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$

$\int_0^1 xe^x dx = \int_0^1 \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}dx = \sum_{n=0}^{\infty}\int_0^1 \frac{x^{n+1}}{n!}dx$

$= \sum_{n=0}^{\infty}\frac{1}{(n+2)n!}$

Q2 $y = g(x)$ satisfies the following differential equation $\boxed{\frac{dy}{dx} = \left(\frac{x+1}{x}\right)y}$.

$\frac{dy}{dx} = xe^x + e^x$

$\frac{dy}{dx} = y + \frac{y}{x} = \left(\frac{x+1}{x}\right)y$

Q3 The above differential equation is $\boxed{\textbf{SEPARABLE}}$ or **NON-SEPARABLE**.

Q4 The domain of $h(p) = \int_1^{\infty}\frac{g(x)}{e^{px}}\,dx$ is $\boxed{p > 1}$.

$h(p) = \int_1^{\infty}\frac{g(x)}{e^{px}}dx = \int_1^{\infty} xe^{(1-p)x}dx = \lim_{c\to\infty} \underbrace{\frac{1}{p-1}xe^{(1-p)x}\Big|_1^c}_{\text{converges if and only if } p>1.} - \frac{1}{1-p}\int_1^{\infty} e^{(1-p)x}dx$

$u = x \quad dv = e^{(1-p)x}dx$

$du = dx \quad v = \frac{1}{1-p}e^{(1-p)x}$

Q5 $g(x)$ has an inverse on the domain $\boxed{(-\infty,-1)\cup(-1,\infty)}$.

$g'(x) = (x+1)e^x$

$g'(x) > 0 \quad \forall x > -1$

$g'(x) < 0 \quad \forall x < -1$

As $e^x > 0\ \forall x \in \mathbb{R}$, the sign of $x+1$ determines if $g(x)$ has an inverse.

Q6 $\int_0^e g^{-1}(x)\,dx = \boxed{e-1}$. (Hint: what is an inverse?)

triangle area $= \frac{1}{2}e^2$

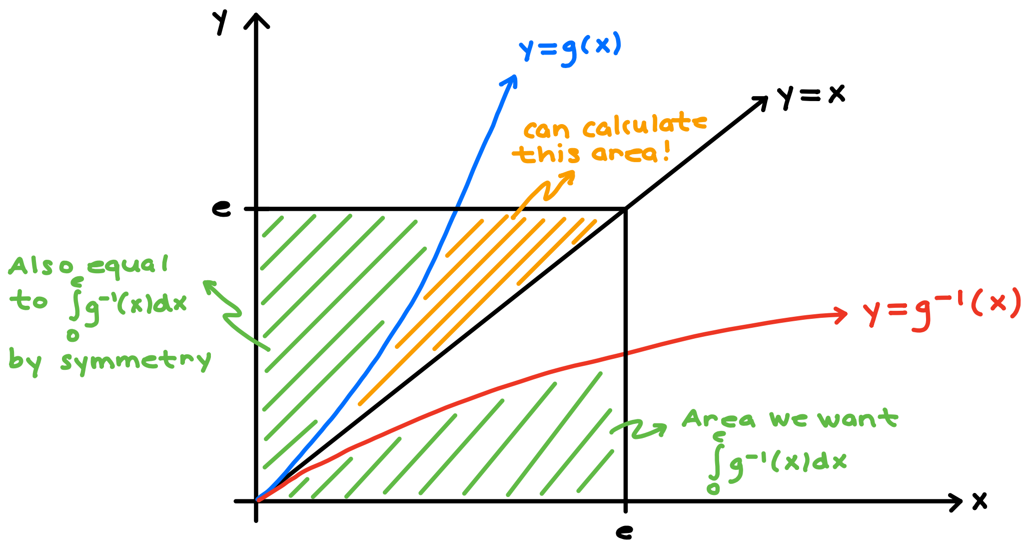
area of orange region $= \int_0^1 g(x) - x\,dx + \int_1^e e - x\,dx$

$= 1 - \frac{1}{2} + e(e-1) - \frac{1}{2}(e^2-1)$

$= 1 + \frac{1}{2}e^2 - e$

$\Rightarrow \int_0^e g^{-1}(x)dx = \frac{1}{2}e^2 - (1 + \frac{1}{2}e^2 - e) = e - 1$

Part 11 *Extra space for work*



End of Exam. Check your work!