Math 1301 Practice Midterm 3 Version 2

Vanderbilt University

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Name: Key

Please do not open the exam until instructed to do so.
You are allowed a one-page (double-sided) formula sheet.
Your instructor may ask to see your formula sheet.
No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

Part 1. (*25 points*) Determine if the following series converge or diverge. Show your work and clearly state any convergence divergence tests you use.

Q1 $\displaystyle\sum_{n=0}^{\infty} \frac{5000^n}{n! \sin^8(n)}$

$$\frac{5000^n}{n!\sin^8(n)} \leq \frac{5000^n}{n!} = a_n$$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{5000 \cdot 5000^n}{(n+1)n!} \cdot \frac{n!}{5000^n} \right|$$

$$= \lim_{n\to\infty} \frac{5000}{n+1} = 0 < 1 \to a_n \text{ converges by ratio test}$$

Direct comparison test yields

$$\sum_{n=1}^{\infty} \frac{5000^n}{n!\sin^8(n)} \text{ converges}$$

Q2 $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n^4 - 3n^2 + 3^n}\right)^{2n}$

$$\lim_{n\to\infty} \sqrt[n]{\left(\frac{n}{n^4-3n^2+3^n}\right)^{2n}} = \lim_{n\to\infty} \frac{n^2}{(n^4-3n^2+3^n)^2} = 0 < 1$$

The root test yields convergence

Q3 $\displaystyle\sum_{n=2}^{\infty} \frac{\ln(\ln(n))}{n\ln(n)}$

Let $f(x) = \dfrac{\ln(\ln(x))}{x\ln(x)}$

$$\int_2^{\infty} f(x)\,dx = \int_2^{\infty} \frac{\ln(\ln(x))}{x\ln(x)}\,dx$$

diverges via integral test

$u = \ln(x)$

$$= \int_{\ln(2)}^{\infty} \frac{\ln(u)}{u}\,du$$

$v = \ln(u)$

$$= \int_{\ln(\ln(2))}^{\infty} v\,dv = \frac{v^2}{2}\Big|_{\ln(\ln(2))}^{\infty} \longrightarrow \infty$$

Q4 $\displaystyle\sum_{n=2}^{\infty} \frac{\cos(n)}{\sqrt{3n-5}}$

Q5 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{n^n}$

Set $a_n = \dfrac{(-1)^n 3^n}{n^n}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{3 \cdot 3^n}{(n+1)^n (n+1)} \cdot \frac{n^n}{3^n}$$

$$= \lim_{n\to\infty} 3 \left(\frac{n}{n+1}\right)^n \frac{1}{n+1} = 0 < 1$$

converges via ratio test

Q6 $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n n}{2n-1}$

Part 2. (*10 points*) Find the interval of convergence for the following power series.

$$p(x) = \sum_{n=2}^{\infty} \frac{(-1)^n}{n\ln(n)}(x-1)^n$$

Part 3 (*10 points*) Determine if the following are true or false. If the statement is false, provide a counter-example; if the statement is true, try to justify your answer.

Q1 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} -a_n$ converge, then $\sum_{n=1}^{\infty} |a_n|$ converges.

False, let $a_n = (-1)^n/n$

Q2 If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges.

False

Q3 If $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| < 1$, then $\sum_{n=1}^{\infty} a_n$ converges.

False, if $\lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| < 1 \Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$

Q4 The sum of the telescoping series $\sum_{n=1}^{\infty}(a_n - a_{n+1})$ is a_1.

False ; $\sum_{n=1}^{\infty}(a_n - a_{n+1}) = a_1 + \lim_{n\to\infty} a_n$

Q5 The 37th partial sum for the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+5}$ is an overestimate.

False

Q6 Every convergent series must pass the ratio test.

False, consider $\sum 1/n^2$

Q7 If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ converges.

Q8 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both converge, then $\sum_{n=1}^{\infty} a_n b_n$ also converges.

False, $a_n = b_n = (-1)/\sqrt{n}$.

Q9 A power series always converges at at least one point.

True, always converges at center

Q10 If the series $\sum_{n=1}^{\infty} a_n$ converges and $a_n > 0$, then $\sum_{n=1}^{\infty} (a_n)^2$ converges.

True

Part 4. (*15 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 The value of $\displaystyle\sum_{n=0}^{\infty} \frac{45^{n+1}}{3^{4n} n!}$ is $\boxed{45e^{45/81}}$

$$\sum_{n=0}^{\infty} \frac{45^{n+1}}{3^{4n}\, n!} = \sum_{n=0}^{\infty} \frac{45 \cdot 45^n}{81^n\, n!} = 45 \sum_{n=0}^{\infty} \frac{(45/81)^n}{n!} = 45e^{45/81}$$

Q2 The number of terms to estimate $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ with an error of less than $\dfrac{1}{100}$ is $\boxed{}$

$$a_0 = \frac{1}{0!}$$

Q3 Use power series to evaluate following limit.

$$\lim_{x\to 0} \frac{\sin(x) - x}{x^3}$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\Rightarrow \sin(x) - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\Rightarrow \frac{\sin(x) - x}{x^3} = \frac{-1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$$

$$\Rightarrow \lim_{x\to 0} \frac{\sin(x) - x}{x^3} = \frac{-1}{3!}$$

Part 5. (*20 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 Find a series representation for π using $\tan^{-1}(x)$.

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} \Rightarrow \tan^{-1}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

$$\tan^{-1}(1) = \pi/4$$

$$\Rightarrow \pi/4 = \tan^{-1}(1) = \sum_{n=1}^{\infty} \frac{(-1)^n 1^{2n+1}}{2n+1} \Rightarrow \pi = \sum_{n=0}^{\infty} \frac{4(-1)^n}{2n+1}$$

Q2 Find the first four terms in the Taylor series centered at $x = 1$ for $g(x) = 5^x$.

$g(x) = 5^x$, $g(1) = 5$

$g'(x) = \ln(5) 5^x$, $g'(1) = \ln(5) 5$

$g''(x) = \ln^2(5) 5^x$, $g''(1) = \ln^2(5) 5$

$g'''(x) = \ln^3(5) 5^x$, $g'''(1) = \ln^3(5) 5$

$$g(x) \sim 5 + \frac{\ln(5)5}{1}(x-1) + \frac{\ln^2(5)5}{2!}(x-1)^2 + \frac{\ln^3(5)5}{3!}(x-1)^3$$

Part 6. (*10 points*) Use power series to evaluate the integral

$$\int \ln(x)\arctan(x)dx.$$

Hint: Use integration by parts first.

$$\int \ln(x)\tan^{-1}(x)\,dx$$

$u = \ln(x) \qquad dv = \tan^{-1}(x)dx$

$du = \frac{dx}{x} \qquad v = x\tan^{-1}(x) - \frac{1}{2}\ln|1+x^2|$

$$= x\ln(x)\tan^{-1}(x) - \frac{1}{2}\ln(x)\ln(1+x^2) - \int \tan^{-1}(x) - \frac{1}{2}\frac{\ln(1+x^2)}{x}\,dx$$

$$\ln(1+x^2) = \sum_{n=1}^{\infty} \frac{(-1)x^{2n}}{n} \Rightarrow \frac{\ln(1+x^2)}{x} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n-1}}{n}$$

Integrating yields

$$\int \frac{\ln(1+x^2)}{x}dx = \sum_{n=1}^{\infty} \frac{(-1)\,x^{2n}}{2n^2}$$

Part 7. (*10 points*) Prove the identity $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$ in the following steps:

Q1 Find the power series representation of $p(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}}$.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$$

$$\sin(\sqrt{x}) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1/2}}{(2n+1)!}$$

$$p(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(2n+1)!}$$

$$= 1 + \frac{-1}{3!}x + \frac{1}{5!}x^2 + \ldots$$

Q2 Find the zeros of $p(x)$.

$$p(x) = \frac{\sin(\sqrt{x})}{\sqrt{x}} = 0 \longleftrightarrow \sin(\sqrt{x}) = 0 \longleftrightarrow x = n^2\pi^2 \; , \; n = 1, 2, 3, 4, \ldots$$

Q3 Use the following fact to prove the result: *For a polynomial* $p(x) = a_n x^n + .. + a_1 x + a_0$ whose roots are $r_1, ..., r_n$,

$$\frac{1}{r_1} + ... + \frac{1}{r_n} = -\frac{a_1}{a_0}.$$

i) $\sum_{n=1}^{\infty} \frac{1}{r_n} = \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$

ii) $-\frac{a_1}{a_0} = -\frac{-1/3!}{1} = \frac{1}{6}$

It follows that

$$\frac{1}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{1}{6} \Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

End of Exam. Check your work!