Math 1301 Practice Midterm 3 Version 1

Vanderbilt University

1 October 2025

Name: Key

Please do not open the exam until instructed to do so.
You are allowed a one-page (double-sided) formula sheet.
Your instructor may ask to see your formula sheet.
No calculators, phones, computers, smart watches, etc. are permitted.

The Vanderbilt Honor Code applies.

Part 1. (*25 points*) Determine if the following series converge or diverge. Show your work and clearly state any convergence divergence tests you use.

Q1 $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}}$

Q2 $\displaystyle\sum_{n=1}^{\infty} \frac{n+2}{\sqrt{n^4+1}}$

Q3 $\displaystyle\sum_{n=3}^{\infty} \frac{\ln(n)}{n^3}$

Q4 $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{(n!)^2}$

set $a_n = \dfrac{n^n}{(n!)^2}$

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(n+1)(n+1)^n}{(n+1)^2(n!)^2} \cdot \frac{(n!)^2}{n^n}\right| = \left(\frac{n+1}{n}\right)^n \frac{1}{n+1}$$

$$\Rightarrow \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 > 1$$

Therefore, the series converges via the ratio test.

Q5 $\displaystyle\sum_{n=3}^{\infty} \frac{1}{(\ln(n))^{\ln(n)}}$

$$\ln(n)^{\ln(n)} = e^{\ln(n)\ln(\ln(n))} = n^{\ln(\ln(n))} > n^2$$

Q6 $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n^5}{5n^5+4n^3+1}\right)$

Compute $\displaystyle\lim_{n\to\infty} \ln\left(\frac{n^5}{5n^5+4n^3+1}\right) = \ln(1/5) \neq 0$

So, diverges via divergence test.

Part 2. (*10 points*) Find the interval of convergence for the following power series.2

$$p(x) = \sum_{n=1}^{\infty} \frac{n^5}{5^n n!}(x-2)^n$$

i) center @ $x = 2$

ii) $\left| \frac{a_{n+1}(x-2)^{n+1}}{a_n (x-2)^n} \right| = \left| \frac{(n+1)^5 \cdot 5^n n!}{5 \cdot 5^n (n+1) n! \, n^5} (x-2) \right|$

$= \frac{(n+1)^5}{5n^6+n^5} |x-2|$

$\Longrightarrow \lim_{n\to\infty} \left| \frac{a_{n+1}(x-2)^{n+1}}{a_n (x-2)^n} \right| = 0$

Therefore, ROC $= \infty$ and IOC $= \mathbb{R}$.

Part 3 (*20 points*) Determine if the following are true or false. If the statement is false, provide a counter-example; if the statement is true, try to justify your answer.

Q1 If $\lim_{n\to\infty} a_n = 0$, then $\sum_{n=1}^{\infty} a_n$ converges.

False, let $a_n = 1/n$.

Q2 If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty}(-1)^n a_n$ also diverges.

False, let $a_n = 1/n$

Q3 If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \frac{\pi}{e}$, then $\sum_{n=1}^{\infty} a_n$ diverges.

True, $\pi/e > 1 \Rightarrow$ ratio test yields convergence

Q4 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty}(a_n + b_n)$ also diverges.

False, set $a_n = (-1)^n$ and $b_n = (-1)^{n-1}$

Q5 If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=1}^{\infty}(a_n)^2$ also converges.

False, consider $a_n = \frac{(-1)^n}{\sqrt{n}}$

Q6 If $\sum_{n=1}^{\infty} a_n < \sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ also diverges.

False, to use Direct Comparison Test to show divergence, you need to bound Σa_n from below

Q7 If $\sum_{n=1}^{\infty} a_n$ conditially converges, then $\sum_{n=1}^{\infty} |a_n|$ must diverge.

True, by definition of conditional convergence.

Q8 If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ both diverge, then $\sum_{n=1}^{\infty} a_n b_n$ also diverges.

False, set $a_n = b_n = 1/n$.

Q9 If $\sum_{n=1}^{\infty} a_n$ diverges, then $\lim_{n\to\infty} a_n = 0$.

False, set $a_n = n$.

Q10 The series $\sum_{n=1}^{\infty} \frac{(-1)^n \cos(n\pi)}{n^2}$ converges by the alternating series test.

False, $\frac{(-1)^n \cos(n\pi)}{n^2} = \frac{1}{n^2} \rightarrow$ converges via p-series

Part 4. (*15 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 The value of $\sum_{n=1}^{\infty} \frac{6^{n+2}}{3^{2n}}$ is $\boxed{72}$

$$\sum_{n=1}^{\infty} \frac{6^{n+2}}{3^{2n}} = \sum_{n=1}^{\infty} \frac{36\ 6^n}{9^n} = \sum_{n=1}^{\infty} 36\left(\frac{2}{3}\right)^n = \frac{36\ ^2/_3}{1-{}^2/_3} = \frac{24}{^1/_3} = 72.$$

Q2 The value of $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ is $\boxed{1}$

$$\frac{1}{n^2+n} = \frac{A}{n} + \frac{B}{1+n}$$

$A=1, \quad B=-1$

$$S_N = \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{1+N}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2+n} := \lim_{N\to\infty} S_N = \lim_{N\to\infty}\left(1 - \frac{1}{1+N}\right) = 1.$$

Q3 The domain of the function $f(p)$ is $\boxed{}$ where $f(p)$ is given by

$$f(p) = \sum_{n=1}^{\infty} \frac{\ln(1+n^p)}{n^p}$$

Part 5. (*20 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 Find a power series representation for $f(x) = \dfrac{x-5}{(1-x)^2}$ and determine its radius of convergence.

Q2 Find the first four terms in the Taylor series centered at $x = 3$ for $g(x) = \sin^2(x)$.

$g(x) = \sin^2(x)$; $g(3) = \sin^2(3)$

$g'(x) = \sin(2x)$; $g'(3) = \sin(6)$

$g''(x) = 2\cos(2x)$; $g''(3) = 2\cos(6)$

$g^{(4)}(x) = -4\sin(2x)$; $g^{(4)}(3) = -4\sin(6)$

Part 6. (*10 points*) Showing work is not required but partial credit may be awarded if you do.

Q1 Find the power series representation of $h(x) = \ln(1 + x^2)$.

$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n \Rightarrow \ln(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

$$\Rightarrow h(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}$$

Q2 Using your solution to Q1, determine the value of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)}$. (*Hint:* $\displaystyle\int_0^1 x^{2n} = \frac{1}{2n+1}$)

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n(2n+1)} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \int_0^1 x^{2n} dx = -1 \int_0^1 \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{n} dx = -\int_0^1 h(x) dx$$

$$\int_0^1 h(x) dx = \int_0^1 \ln(1+x^2) dx$$

$u = \ln(1+x^2)$ $\quad dv = dx$

$du = \frac{2x}{1+x^2} dx$ $\quad v = x$

$$= \underbrace{x \ln(1+x^2) \Big|_0^1}_{= \ln(2)} - \int_0^1 \frac{2x^2}{1+x^2} dx$$

$$\int_0^1 \frac{2x^2}{1+x^2} dx = 2 \int_0^1 \frac{x^2+1-1}{1+x^2} dx$$

$$= 2\left[\int_0^1 1 \, dx - \int_0^1 \frac{dx}{1+x^2}\right]$$

$$= 2 - 2\tan^{-1}(1)$$

$$= 2 - \pi/2$$

$\Rightarrow$

End of Exam. Check your work!