Final Exam Challenge Problems

Here are some problems that are beyond the scope of the final exam, but use concepts from calculus 2. If you are comfortable with the material, try these to see how you will integrate this class with other types of problem. Plus, they are cool.

1. Define the function $\Gamma(x) = \displaystyle\int_0^\infty t^{x-1}e^{-t}\,dt$. This function is known to mathematicians as the **gamma function**.

   (a) Calculate $\Gamma(1)$

   (b) Use integration by parts on $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$.

   (c) Use parts (a) and (b) to conclude that $\Gamma(x) = (x-1)!$.

2. Define the function $\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$. This function is known by mathematicians as the **Riemann zeta function**. We call this function the **p-series** in Math 1301.

   (a) Use the integral test to find the domain of $\zeta(x)$.

   (b) Prove the formula $\zeta(x)\Gamma(x) = \int_0^\infty \frac{u^{x-1}}{e^x - 1} du$ in the following steps.

      i. Make the substitution $t = nu$, where n is a constant, to $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\, dt$ and show this yields $\frac{1}{n^x}\Gamma(x) = \int_0^\infty u^{x-1}e^{-nu}\, du$

      ii. Apply $\sum_{n=1}^{\infty}$ on the above result to show that $\zeta(x)\Gamma(x) = \int_0^\infty u^{x-1} \sum_{n=1}^{\infty} e^{-nu}\, du$

      iii. Use the geometric series formula on $\sum_{n=1}^{\infty} e^{-nu}$ to show $\zeta(x)\Gamma(x) = \int_0^\infty \frac{u^{x-1}}{e^x - 1} du$

3. Evaluate the integral $-\displaystyle\int_0^1 \frac{\ln(1-x)}{x}\,dx$ via the power series representation of $\ln(1-x)$.

4. In 1697, Johann Bernoulli proved the following integral result: $\displaystyle\int_0^1 \frac{1}{x^x}\,dx = \sum_{n=1}^{\infty} \frac{1}{n^n}$.

   We will prove this in the following steps.

   (a) Using the gamma function and the substitution $t = -\ln(x)$, show that
   $$\int_0^1 x^n \ln^n(x)\,dx = (-1)^n \frac{\Gamma(n+1)}{(n+1)^{n+1}} = (-1)^n \frac{n!}{(n+1)^{n+1}}$$

   (b) Write $\dfrac{1}{x^x}$ as a power series by writing $x^{-x} = e^{-x\ln(x)}$.

   (c) Integrating the power series you found above and using part (a), show the result.

5. Consider the sequence $\gamma_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n)$. We are going to show this sequence converges in the following steps.

   (a) Show that γ_n is monotonically decreasing.

   (b) Show that $\gamma_n > 0$.

   (c) Use the monotone convergence theorem to show that γ_n converges.

   This constant was discovered by Euler in 1734 and is known by mathematicians as the **Euler-Mascheroni constant** and is denoted by γ. This constant has been elusive to mathematicians for almost three centuries. For example, it is not known if γ is rational!

6. A common approximation to π is $\dfrac{22}{7}$. We have

$$\pi = 3.14159... \qquad \text{and} \qquad \frac{22}{7} = 3.1428.$$

To show that $\pi < \dfrac{22}{7}$, evaluate the integral $\displaystyle\int_0^1 \frac{x^4(1-x)^4}{(x+1)^2}\,dx$.

7. Solve the differential equation $\dfrac{dy}{dx} = \dfrac{x+y}{x-y}$ via the substitution $v = \dfrac{y}{x}$ to make the original differential equation separable.

8. Let $f(x)$ be a continuously differentiable function such that $f(a) = f(b) = 0$. If

$$\int_a^b (f(x))^2 \, dx = 1,$$

show that

$$\int_a^b x f(x) f'(x) dx = -\frac{1}{2}$$

9. Let f be a continuous differentiable function such that

$$\lim_{x\to\infty} (f(x) + f'(x)) = L.$$

Use L'Hopital's rule to show that $\lim_{x\to\infty} f(x) = L$ and $\lim_{x\to\infty} f'(x) = 0$.